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# A Study on Rocking and Overturning of Rectangular Column

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**Abstract**

The rocking motion and overturning of rectangular columns have been studied in order to estimate earthquake intensity in the former investigations. The overturning condition of a column was considered by some ways of thinking in the past. One is the so-called static overturning condition, and this gives only a necessary condition as well known. The others are based on dynamical point of view and consider the overturning of a column caused by sinusoidal excitation by the method based on energy or the use of periodic solution, where a quarter wave of excitation is used in the former method and infinite excitation is used in the latter.

But, in practice, the transient rocking motion of a column caused by several waves excitation is important for the discussion of overturning. This report describes the similarity laws on overturning and some features of transient rocking motion of a column caused by infinite or finite sinusoidal excitation, and, as an example, the overturning diagram for 3-wave excitation is considered.

## 1. Introduction

Rocking and overturning of rigid bodies are the typical phenomena to be observed during earthquake and have been studied in connection with the estimation of maximum ground acceleration by observing the overturning of grave-stones. There are many factors which have effects on the overturning phenomena and some of them have been made clear in the former investigations. Many problems, however, are remained, and the estimation of dynamic effects is one of them.

In this report the author studied on this problem using a simple rocking model. It is easy to obtain the numerical solution of a simple rocking motion for individual columns, but, it is difficult to give the general discussion on overturning by reason that the equation of motion is changed at collision of a column to the base. Hence, some approximate or numerical methods are required, and then, at first the type of excitation is to be assigned, and secondly the similarity laws must be established for general application of the results obtained. In this report, the sinusoidal excitation

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with stationary amplitude is assumed, and similarity laws and some features of rocking and overturning will be described.

## 2. Equation of motion and models for numerical simulation

The rocking motion of a rectangular column which is submitted to horizontal ground motion $\ddot{u}$ is given by

$$\ddot{\theta} = n^2\ddot{u}\cos(\theta\pm\alpha)+n^2 g\sin(\theta\pm\alpha) \qquad \theta \lesseqgtr 0 \tag{1}$$

in which

$$n^2 = \frac{Mr}{I_G+Mr^2} = \frac{3}{4r} \tag{2}$$

where g is gravity, $I_G$ the moment of inertia at G, $M$ the mass of the column (Fig. 1). In most cases, $\alpha$ and $\theta$ are relatively small values, then the simplified equation

$$\ddot{\theta} = n^2\ddot{u}+n^2 g(\theta\pm\alpha) \qquad \theta \lesseqgtr 0 \tag{3}$$

is often used. In this report, Eq. (3) is mainly used, but Eq. (1) is also used for numerical simulation. Eq. (3) represents the forced vibration of the system which has the discontinuity of the restoring force at $\theta=0$.

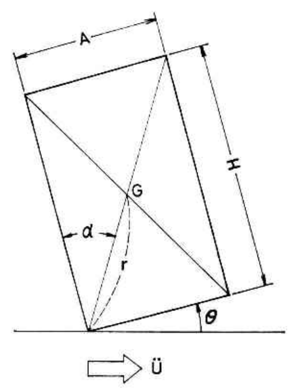


**Fig. 1** Rocking model of a column

Table 1 shows the models for numerical simulation. The models A-1, B-1 and C-1 which have

**Table 1** The models for numerical simulation

| Type | | $A$ cm | $H$ cm | $\alpha$ deg. | $\lambda$ sec$^{-1}$ | $a_S$ gal |
|---|---|---|---|---|---|---|
| A | 1 | 10 | 40 | 14 | 5.97 | 245 |
| | 2 | 20 | 80 | 14 | 4.22 | 245 |
| | 3 | 50 | 200 | 14 | 2.67 | 245 |
| B | 1 | 10 | 30 | 18.4 | 6.82 | 327 |
| | 2 | 20 | 60 | 18.4 | 4.82 | 327 |
| | 3 | 50 | 150 | 18.4 | 3.05 | 327 |
| C | 1 | 10 | 20 | 26.6 | 8.11 | 490 |
| | 2 | 20 | 40 | 26.6 | 5.73 | 490 |
| | 3 | 50 | 100 | 26.6 | 3.63 | 490 |

different values of angle $\alpha$ (or the ratio of width to height) are basic models, and the others are similar models with two or five times large size as the basic models. The value of $a_S$ is the minimum static acceleration for overturning, and the scale factor $\lambda$ is given by

$$\lambda = n\sqrt{g} \tag{4}$$

which appears in the solution of Eq. (3) and has the dimension of frequency. It can be seen from Eq. (2) that the larger the column is, the smaller value takes $\lambda$. Three wooden columns for basic models are used in the following section as practical models for estimation of damping ratio.

## 3. Damped free rocking motion and its phase-plane trajectory

In this section, the overturning condition and the damping of free rocking motion are described. For this purpose, the phase-plane trajectory of motion is useful. The equation of free rocking motion is given by

$$\ddot{\theta} = \lambda^2(\theta \pm \alpha) \qquad \theta \lessgtr 0 \tag{5}$$

Eliminating the time $t$ of Eq. (5), the phase-plane trajectory is obtained as follows:

$$\lambda^2(\alpha - |\theta|)^2 - \dot{\theta}^2 = C \tag{6}$$

where $C$ is an arbitrary constant given by initial conditions. The trajectory (6) with various values of $C$ is represented by curves in $\theta$-$\dot{\theta}$ plane as Fig. 2, and its tangent changes discontinuously at $\theta=0$. The phase-plane is separated into some stable (non-overturning) or unstable (overturning) regions by the linear trajectory corresponding with $C=0$ (chain line in Fig. 2).

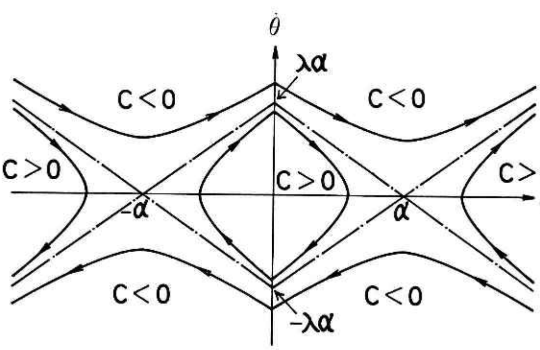


**Fig. 2** Phase-plane trajectory of free rocking motion

There is no damping factor in Eq. (5), but in practice we can see the decrease of free rocking motion. There is an old way of thinking on such a decrease which is based on conservation of angular momentum at changing the center of rotation, and recently a further consideration on collision has been done (Mochizuki & Kobayashi, 1976). There are many practical factors of energy loss of rocking motion, for example, the friction of rolling or slipping, the energy dissipation by collision, etc., but these are troublesome to be estimated quantitatively. Then, for simplicity, it is supposed in this report that the energy loss is represented by only the decrease of angular velocity at collision, that is,

$$\dot{\theta} \longrightarrow \delta\dot{\theta} \text{ at } \theta = 0 \qquad 0 < \delta \leqq 1 \tag{7}$$

in which $\delta$ is the damping ratio, and is supposed to have a constant value. Using Eq. (7), the phase-plane trajectory of damped free rocking motion can be derived. Let $C_0$ be the initial trajectory in stable region and $C_i$ that after $i$ times passing of

$\theta=0$, the following relation is obtained:

$$C_i = \delta^{2i}C_0+(1-\delta^{2i})\lambda^2\alpha^2 \tag{8}$$

A set of the curves $C_0$, $C_1$, $C_2$, etc. connected at $\theta$-axis gives the trajectory of damped free rocking motion with given initial conditions.

The value of $\delta$ may be estimated by measurement or calculation from conservation of angular momentum, and hitherto some comparisons of measured and caltulated value have been done. The author also has compared two estimations using the wooden models A-1, B-1 and C-1. For a simple estimation from experiment, the trajectory in the stable region of the phase-plane is considered. Let $T_i$ be the time of $i$-th passing of $\theta=0$, $C_i$ the trajectory from $T_i$ to $T_{i+1}$, then the time required for the pass of the trajectory $C_i$ is given by

$$T_{i+1}-T_i = \frac{2}{\lambda}\ln\left[\frac{\lambda}{\sqrt{C_i}}(\alpha+\sqrt{\alpha^2-C_i/\lambda^2})\right] \tag{9}$$

Eliminating $C_0$, $C_i$ and $\alpha$ in Eqs. (8) and (9), the damping ratio is given by

$$\delta = \left(\frac{E_{i+1}-1}{E_{i+1}+1}\right)\Big/\left(\frac{E_i-1}{E_i+1}\right) \quad \text{where } E_i = \exp\left[\lambda(T_{i+1}-T_i)\right] \tag{10}$$

Then the value of $\delta$ can be estimated by measurement of $T_i$ only. Table 2 shows the comparison of the measured and the calculated value. It can be seen that the two estimations almost agree in this case and the damping is considerably small. Like this, the damping ratio generally depends on the shape or size of the column and the property of its base, but in this report, for simplicity, it is supposed to be constant independent on them.

**Table 2** Damping ratio of three wooden models

| Model | Measurement | | | | | | | Theoretical |
|---|---|---|---|---|---|---|---|---|
| | $i=1$ | 2 | 3 | 4 | 5 | 6 | Average | |
| A-1 | 0.89 | 0.99 | 0.89 | 0.98 | 0.90 | 0.99 | 0.94 | 0.91 |
| B-1 | 0.89 | 0.89 | 0.89 | 0.85 | 0.89 | 0.84 | 0.88 | 0.85 |
| C-1 | 0.80 | 0.77 | 0.80 | 0.75 | — | — | 0.79 | 0.7 |

Considering the initial trajectory and its change at $\theta=0$, the condition of stability for damped free rocking motion is obtained as follows:

$$\left.\begin{array}{r} C_0>0 \text{ and } |\theta_0|<\alpha \\ \text{or } \left(1-\frac{1}{\delta^2}\right)\lambda^2\alpha^2<C_0<0 \end{array}\right\} \tag{11}$$

where $\theta_0$, $\dot{\theta}_0$ are the initial angle and angular velocity, respectively, and $C_0$ the initial trajectory given by Eq. (6). The condition of overturning is given by the reverse of the above.

## 4. Rocking and overturning by horizontal excitation

There are four parameters in forced rocking motion caused by sinusoidal excitation, that is, the angle (or the ratio of width to height) and the size of a column, and the acceleration amplitude and frequency of excitation, which are fundamental for discussion of overturning. Then, it is the first problem to be considered whether the overturning condition of columns can be expressed by a diagram with these parameters combined.

### (1) General considerations

Substituting the horizontal excitation $\ddot{u}=a\sin(\omega t+\beta)$ beginning at $t=-\beta/\omega$ into Eq. (3) and supposing that the column will begin to rock at $t=0$, the general solution is given by

$$\theta = \pm\alpha + Ae^{\lambda t} + Be^{-\lambda t} - C\sin(\omega t+\beta) \qquad \theta \gtrless 0 \tag{12}$$

in which

$$C = \frac{\lambda^2\alpha}{(\lambda^2+\omega^2)\sin\beta} \tag{13}$$

where $A$ and $B$ are arbitrary constants to be determined from initial conditions, $\beta$ the phase difference resulting from the time difference between the beginnings of excitation and rocking motion. From the condition of motion beginning, $\beta$ is given by

$$\sin\beta = \frac{g\alpha}{a} \tag{14}$$

For the later discussion, $k$ is defined as follows:

$$k = \frac{a}{g\alpha}\left(=\frac{1}{\sin\beta}\right) \tag{15}$$

The solution (12) contains the divergent term $e^{\lambda t}$, though the column can be stable because of the switching of equation at $\theta=0$. If a solution does not take zero, there will be no collision between the column and the base (this collision also will be called rocking in this report) and the solution will diverge to infinity with time, and this will represent the overturning by forced rocking motion.

The solution after $n$ times rocking is expressed by

$$\theta_n = (-1)^n\alpha + A_n e^{\lambda t} + B_n e^{-\lambda t} - C\sin(\omega t+\beta) \tag{16}$$

where

$$\left.\begin{aligned} A_{n+1} &= A_n + (-1)^n\alpha e^{-\lambda t_{n+1}} - \frac{1-\delta}{2\lambda}e^{-\lambda t_{n+1}}\theta_n(t_{n+1}) \\ B_{n+1} &= B_n + (-1)^n\alpha e^{\lambda t_{n+1}} + \frac{1-\delta}{2\lambda}e^{\lambda t_{n+1}}\theta_n(t_{n+1}) \end{aligned}\right\} \tag{17}$$

$$n = 0, 1, 2, \cdots$$

in which $\delta$ is the damping ratio described in the previous section and $t_{n+1}(>t_n)$ the time at which the solution $\theta_n$ takes zero. When $\delta=1$, the third term of Eq. (17) dissapears. Eq. (16) expresses the solution in $\theta>0$ for $n=0,2,4,\cdots$ and that in $\theta<0$ for $n=1,3,\cdots$. If the column will repeat some times rocking, the overall solution, and consequently, the coefficient series $\{A_n\}$, $\{B_n\}(n=0,1,2,\cdots)$ can be obtained. Substituting the initial conditions $\theta=0$ and $\dot{\theta}=0$ at $t=0$ into Eq. (16) and changing, the expressions for $A_0$ and $B_0$ are obtained as follows:

$$\left.\begin{matrix} A_0 \\ B_0 \end{matrix}\right\} = \frac{\omega^2\alpha}{2(\lambda^2+\omega^2)}\left(-1\pm\frac{\omega_0}{\omega}\right) \tag{18}$$

where

$$\omega_0 = \lambda \cot \beta \tag{19}$$

At first, the case of infinite excitation is to be considered. Generally we can say that the condition of non-overturning is the presence of infinite series $\{A_n\}$, $\{B_n\}$ and $\{t_n\}$ $(n=0,1,2,\cdots)$. In the case of $\delta=1$, this statement is converted into a more simple expression as the following. Namely, from Eq. (17), the following characteristics of the series $\{A_n\}$, $\{B_n\}$ are obtained:

$$\begin{matrix} A_0<A_2<\cdots<A_{2n}<\cdots<A_{2n+1}<\cdots<A_3<A_1 \\ \cdots B_4<B_2<B_0<B_1<B_3<\cdots \end{matrix} \tag{20}$$

If the overall solution is stable, it can be seen that $A_n\to0(n\to\infty)$ from Eq. (17). Then the series $\{A_n\}$ of stable rocking motion satisfies

$$A_0<A_2<\cdots<A_{2n}<\cdots<0<\cdots<A_{2n+1}<\cdots<A_3<A_1 \tag{21}$$

or

$$\left.\begin{matrix} A_n<0 \text{ when } n \text{ is even} \\ A_n>0 \text{ when } n \text{ is odd} \end{matrix}\right\} \tag{22}$$

On the contrary, if the coefficient $A_n$ satisfies the condition (22) successively beginning from $n=0$, the rocking motion will be infinitely repeated and the overall solution is stable. Thus the relation (22) gives the condition of stability of rocking motion in the case of infinite excitation and non-damping. From this result, if the condition (22) does not hold for a value of $n$, the column will overturn. At this time, $\{A_n\}$ will be a finite series as follows:

$$\{A_n\} = A_0, A_1, A_2, \cdots, A_M, \cdots, A_{M+m} \tag{23}$$

If all terms of the series (23) satisfy the condition (22), the term $A_{M+m+1}$ also must be found, therefore a first term $A_M$ which does not agree with the condition (22) must be present. When $M$ is even, it can be seen from the relation (20) that all terms after $A_M$ are positive, then the column will overturn to the direction of its first motion. Similarly it will overturn to the opposite direction of that when $M$ is odd.

When $\delta \neq 1$, the relation (22) is not necessarily a condition to be required for non-overturning because of the presence of the third term in Eq. (17), but it gives a sufficient condition for non-overturning.

Next, the case of finite excitation is considered. Let $T$ be the time of end of excitation and $t_M(\leqq T)$ the time of rocking nearest to $T$, the solution in the range of $t \geqq t_M$ is given by

$$\theta_M(t) = (-1)^M \alpha + A_M e^{\lambda t} + B_M e^{-\lambda t} - C \sin(\omega t + \beta) \tag{24}$$

In general we can say that the stability of the column can be determined by only the state at $t=T$. Let $N$ (positive and real) be the wave number of excitation, the rocking angle $\theta_M$ and the angular velocity $\dot{\theta}_M$ at $t=T$ are rewritten as follows:

$$\left.\begin{aligned} \theta_M(T) &= (-1)^M \alpha + A_M \exp\left[\frac{\lambda}{\omega}(2N\pi - \theta)\right] + B_M \exp\left[\frac{\lambda}{\omega}(2N\pi - \beta)\right] - C \sin 2N\pi \\ \theta_M(T) &= \lambda\left\{A_M \exp\left[\frac{\lambda}{\omega}(2N\pi - \beta)\right] - B_M \exp\left[\frac{\lambda}{\omega}(2N\pi - \theta)\right] - \frac{\omega}{\lambda} C \cos 2N\pi\right\} \end{aligned}\right\} \tag{25}$$

From Eq. (11) and above, the stability condition of free rocking motion after $t=T$ is given by

$$\left.\begin{aligned} & |\theta_M(T)| \leqq \alpha \text{ and } \{\alpha - |\theta_M(T)|\}^2 - \{\dot{\theta}_M(T)/\lambda\}^2 > 0 \\ \text{or} \quad & \theta_M(T)\theta_M(T) < 0 \text{ and } \left(1 - \frac{1}{\delta^2}\right)\alpha^2 < \{\alpha - |\theta_M(T)|\}^2 - \{\theta_M(T)/\lambda\}^2 < 0 \end{aligned}\right\} \tag{26}$$

If the above does not hold, the column will have overturned or be overturning at $t=T$.

From the considerations above described, the similarity laws on stability or overturning of columns can be derived. As previously mentioned, the four parameters have effects on stability of columns, but these effects can be reduced to a relation of two variables only. Namely, from Eqs. (13), (18), $A_0$, $B_0$ and $C$ are all expressed in the following form:

$$\alpha \times F\left(\frac{\omega}{\lambda}, k\right) \tag{27}$$

where $F$ is an arbitrary function of $\omega/\lambda$ and $k$. Write $x=\omega t$, then the $n$-th solution $\theta_n$ is expressed by the following:

$$\left.\begin{aligned} \theta_n(t) &= (-1)^n \alpha + A_n \exp\left[\frac{\lambda}{\omega}x\right] + B_n \exp\left[-\frac{\lambda}{\omega}x\right] - C \sin(x + \beta) \\ \theta_n(t) &= \lambda\left\{A_n \exp\left[\frac{\lambda}{\omega}x\right] - B_n \exp\left[-\frac{\lambda}{\omega}x\right] - \frac{\omega}{\lambda} C \cos(x + \beta)\right\} \end{aligned}\right\} \tag{28}$$

From these equations, it can be seen that every term of $\{A_n\}$ and $\{B_n\}$ is expressed by the form (27) and that the presence of a zero point $t(>t_n)$ of the $n$-th solution $\theta_n(t)$ is determined only from the values of $\omega/\lambda$ and $k$. Then the presence of infinite series $\{A_n\}$, $\{B_n\}$ and $\{t_n\}$ for infinite excitation depends on $\omega/\lambda$ and $k$ only. Therefore the stability condition of columns submitted to infinite excitation is represented by a region in $\omega/\lambda$-$k$ plane, and this region depends on $\delta$.

In the case of finite excitation, the same result is derived from Eq. (26) if $N$ and $\delta$ are constant. At this time, $M$ in Eq. (25) is also determined only from $\omega/\lambda$ and $k$. It is to be noted that the result is not true if the exciting time is constant.

From the above conclusion, when the wave number of excitation $N$ and the damping ratio $\delta$ are constant, the following similarity laws on overturning or stability hold:

(1) Since $k=\dfrac{a}{g\alpha}\simeq\dfrac{a}{g(A/H)}$, $S$ times the acceleration amplitude of excitation is equivalent to $1/S$ times the angle $\alpha$ (nearly equal to the ratio of width to height) of a column.

(2) Since $\lambda=\sqrt{3g/4r}$, $S$ times the frequency of excitation is equivalent to $S^2$ times the size of a column.

These similarity laws have been pointed out from the consideration of periodic solution or experimental results in the former investigations, but here, a transient rocking motion of a column with damping ratio $\delta$ by $N$-wave excitation has been considered. These results are approximately true because they are based on Eq. (3).

Figs. 3 and 4 show the responses of the models A-1, 2 and B-1, 3 for the common values of $\omega/\lambda$ and $k$ in the case of $N=3$. In these figures the angle $\alpha$ of each model is shown in a same scale and the damping ratio $\delta$ is taken as 1 or 0.9. The cases of overturning are shown in Figs. 4-1, 2 and those of non-overturning in Figs. 3-1, 2. The difference of the overturning direction in Fig. 4-1 and Fig. 4-2 shows an effect

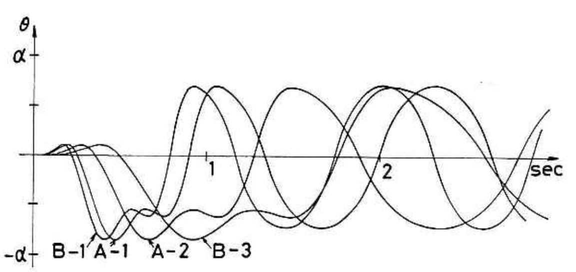


**Fig. 3. 1** Similarity of responses for four models in the case of non-overturning ($k=1.5$, $\omega/\lambda=2.9$, $\delta=1$)

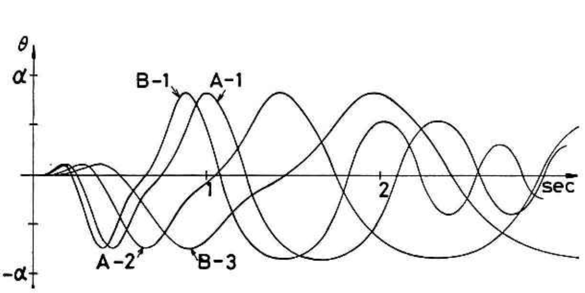


**Fig. 3. 2** Similarity of responses for four models in the case of non-overturning ($k=1.5$, $\omega/\lambda=2.9$, $\delta=0.9$)

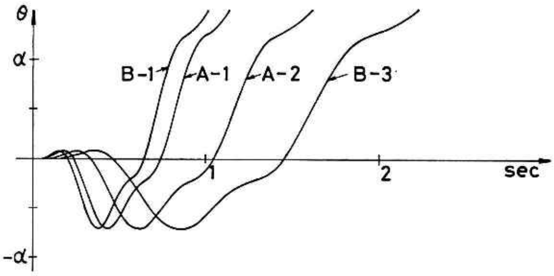


**Fig. 4. 1** Similarity of responses for four models in the case of overturning ($k=1.5$, $\omega/\lambda=3.1$, $\delta=1$)

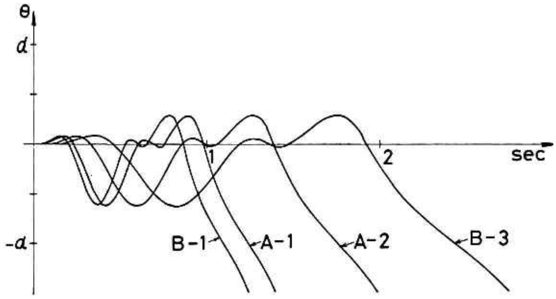


**Fig. 4. 2** Similarity of responses for four models in the case of overturning ($k=1.5$, $\omega/\lambda=3.1$, $\delta=0.9$)

of damping. From these figures it can be seen that the responses of these models agree precisely exclusive of the difference of $\theta$- and time scale.

It seems to be impossible to obtain the region of stability or overturning theoretically because the analytical solution of transcendental equation is necessary for that purpose. Hence, some particular cases based on approximation or numerical calculation will be described in the later parts.

(2) **Quasi-static overturning of a column**

The case that the stability condition (22) for infinite excitation does not hold for the first term ($n=0$), that is, $A_0 \geqq 0$ gives a special case of overturning. For this case the column will overturn as previously described, but furthermore, it can be seen that the column will overturn without rocking after its first motion. Namely, the part $\alpha + A_0 e^{\lambda t} + B_0 e^{-\lambda t}$ in the first solution $\theta_0$ is increasing with $t$ because that $A_0 \geqq 0 > B_0$. Since the remaining part $-C\sin(\omega t+\beta)$ in $\theta_0$ is periodic, if $\theta_0>0$ for $0<t<(\pi/2-\beta)/\omega$, then $\theta_0>0$ for any value of $t>0$. Expanding $\theta_0$ into a Taylor series and considering $\theta_0=\dot{\theta}_0=\ddot{\theta}_0=0$ at $t=0$, the following equation is obtained:

$$\theta_0 = \frac{1}{6}\dddot{\theta}_0(\eta)t^3 \qquad 0<\eta<t \tag{29}$$

Using Eq. (16)

$$\dddot{\theta}_0(\eta) = \lambda^3 A_0 e^{\lambda\eta} - \lambda^3 B_0 e^{-\lambda\eta} + \omega^3 C \cos(\omega\eta+\beta) \tag{30}$$

From Eqs. (29) and (30), it can be seen that $\theta_0>0$ for $0<t<(\pi/2-\beta)/\omega$. Thus the solution $\theta_0$ will diverge without taking zero, namely, the column will overturn without rocking.

Now, the condition that $A_0 \geqq 0$ is equivalent to the relation

$$\omega \leqq \omega_0 \tag{31}$$

Solving the above for the acceleration amplitude $a$, the following relation is obtained:

$$a \geqq g\alpha\sqrt{1+(\omega/\lambda)^2} \tag{32}$$

When $\omega=0$, this agrees with the so-called static condition of overturning based on Eq. (3). Fig. 5 shows the relation (32) for the models A-1, B-1 and C-1. As is shown in this figure, the required acceleration becomes very large for high frequency. Then, this kind of overturning is possible to occur in the case of excitation with low frequency and large amplitude. In that meaning it may be called a quasi-static overturning. Since the value of right hand side of (32) becomes smaller with increasing of $\lambda$ and decreasing of $\alpha$, then such an overturning may be the more probable for slender and small columns. In this case the damping ratio need not to be considered because of no collision. Fig. 6 shows the response of the model A-1 for the excitation with frequency of 2 Hz and gradually increased acceleration amplitude. In this computation the non-linerar equation (1) was used. It can be seen that the minimum acceleration amplitude for non-rocking overturning almost agrees with the value of Fig. 5.

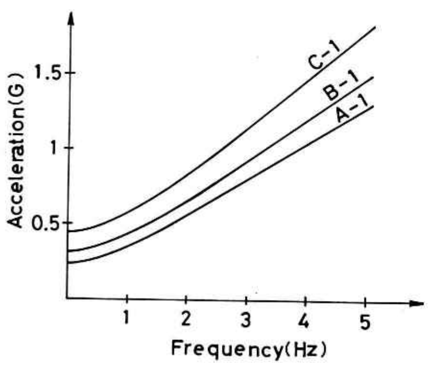


**Fig. 5** The range of quasi-static overturning for three models

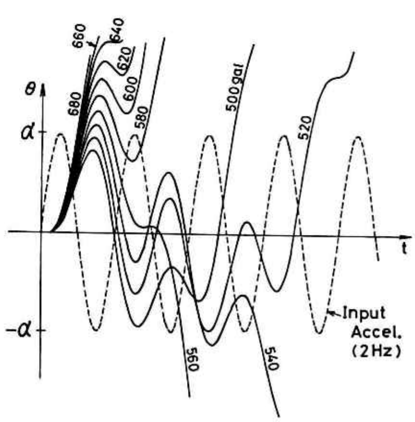


**Fig. 6** Response of the model **A-1** depending on input acceleration amplitude (case of 2 Hz)

In the beginning, the excitation was supposed to be infinite, but clearly it is not necessary. The wave number of excitation required for a quasi-static overturning can be estimated by use of the first of the condition (11). Let $T$ be the time of the end of excitation and $N+\varepsilon$ ($N$ is positive integer and $0 \leqq \varepsilon < 1$) be the wave number, then the non-rocking overturning condition from the free rocking motion after $t=T$ is given by

$$\left.\begin{array}{ll} & A_0 e^{\lambda T}+B_0 e^{-\lambda T}>C \sin 2\varepsilon\pi \\ \text{or} & \{\omega C \cos 2\varepsilon\pi+\lambda C \sin 2\varepsilon\pi-2\lambda A_0 e^{\lambda T}\}\times\{\omega C \cos 2\varepsilon\pi-\lambda C \sin 2\varepsilon\pi+2\lambda B_0 e^{-\lambda T}\}>0 \end{array}\right\} \tag{33}$$

Since $A_0 \geqq 0 > B_0$, the second of (33) holds regardless of $N$ if

$$\frac{1}{2}-\frac{1}{2\pi}\tan^{-1}\frac{\omega}{\lambda}<\varepsilon<\frac{1}{2}+\frac{1}{2\pi}\tan^{-1}\frac{\omega}{\lambda} \tag{34}$$

Thus one half wave of excitation is sufficient for overturning, and in the case of large size column and high frequency of excitation, about a quarter wave is sufficient.

(3) **Quasi-stable rocking motion of a column**

If $\omega \geqq \omega_0$, one or several times rocking will occur. Here, an approximate estimation on the safety of a column for the case of $\omega \geqq \omega_0$ and $\delta=1$ is considered. In this case, from Eq. (17), the following is obtained:

$$\cdots<B_4<B_2<B_0<0<B_1<B_3<\cdots \tag{35}$$

Then the stability condition (22) is equivalent to the relation $A_n B_n>0$. Hence, the column will rock at least $M$ times if

$$A_n B_n>0 \qquad n=0,1,2,\cdots,M-1 \tag{36}$$

Using Eq. (16) and that $\theta_n(t_{n+1})=0(n=0,1,2,\cdots)$, $A_n B_n$ is given by

$$A_n B_n = A_0 B_0+\alpha C\sum_{i=0}^{n-1}(-1)^i \sin(\omega t_{i+1}+\beta) \tag{37}$$

Since a further estimation of the above is difficult because the value of $t_n$ is not analytically obtained, then a sufficient condition for the relation (36) is considered here. Namely, supposing that $(-1)^i \sin(\omega t_{i+1}+\beta)=-1$ and substituting Eq. (37) into (36), the following relation is obtained:

$$A_0 B_0>(M-1)\alpha C \tag{38}$$

Substitution of Eq. (18) gives

$$\frac{\omega}{\lambda}>\sqrt{\frac{k^2+(M-1)k-1+\sqrt{\{k^2+(M-1)k-1\}^2+4(M-1)k}}{2}} \tag{39}$$

This gives one of the sufficient conditions in order to rock more than $M$ times for infinite excitation. In that meaning it may be called quasi-stable rocking motion with an index $M$ representing the number of rocking.

The relation (39) for various values of $M$ is almost liner in $\omega/\lambda$-$k$ plane except for $M=1$ as is shown in Fig. 7. In the right side region of the curve for each $M$ the column will not overturn before $M$ times rocking. This figure can be used for rough estimation of the safety of columns. For example, in the case of the same values of $\omega$ and $k$, the smaller $\lambda$ is, the larger becomes $\omega/\lambda$; then the larger $M$ can be taken. Namely, when similar columns are submitted to the same excitation, the larger column has the more stability. But this is not always an exact estimation as will be shown later.

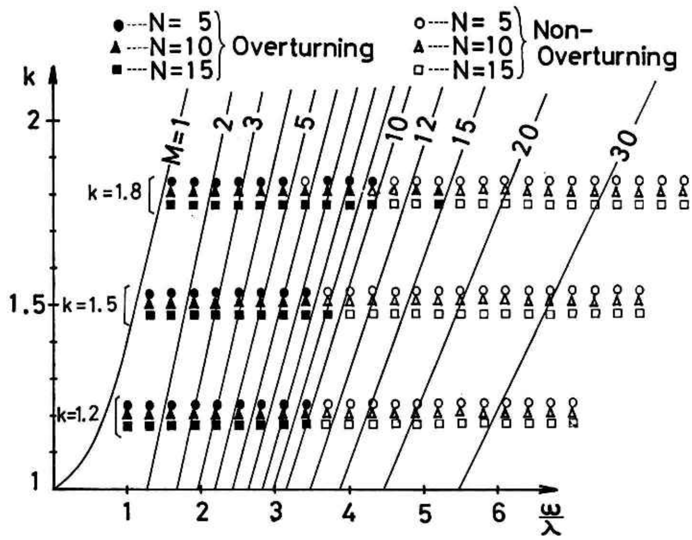


**Fig. 7** A diagram for quasi-stable rocking motion, and numerical simulations in the case of $N$-waves excitation

Now, in order to have an approximate estimation for the case of finite excitation, let us assume that

(1) if the column has a rocking after $t=T$ for infinite excitation, it will not overturn though the excitation ends at $t=T$, and that

(2) the number of rocking is, in average, less than two per period of excitation. These assumptions are considered to be nearly reasonable for a comparative long excitation. From these assumptions it is derived that the column submitted to $N$-wave excitation ($N$: interger) will quasi-statically overturn in the area of $M\leqq 1$ in Fig. 7, will not overturn in $M>2N+1$, and both cases are possible in $1<M\leqq 2N+1$. As an example, the results of simulations using the model A-1 for $N=5$, 10, 15 are plotted in Fig. 7, in which the value of $k$ is taken as 1, 1.5 and 1.8. It can be seen from this figure that the above estimation roughly applies and that it is rather overestimation for safety when $N$ is large. This figure also shows the well-known tendency that the overturning acceleration is nearly equal to the static overturning acceleration in the range of lower frequency, but it rapidly becomes larger as the frequency increases.

**(4) The overturning diagram for three-wave excitation**

Here the case of 3-wave excitation is considered as an example of dynamic overturning diagram in $\omega/\lambda$-$k$ plane. In this case, the numerical method is used and the value of $\delta$ is taken as 1.

Fig. 8 shows the result of computation in the range of $1.15\leqq k\leqq 1.9$. Like this figure, the overturning condition for $N$-wave excitation is generally given by an intricate area with many branches. The overturning for smaller value of $\lambda$ and the non-overturning for its larger value corresponding with the same values of $\omega$ and $k$ can be found in this figure. Namely, when the similar columns are submitted to the same excitation, it is possible that the larger column overturns and the smaller does not.

In order to apply this diagram to individual columns with various sizes, it is convenient to change it into a diagram in $f$-$K$ plane, where $f$ is the frequency of excitation and $K$ the usual scale of earthquake intensity $(a/g)$. For simplicity, the linear approximation of the curve in Fig. 8 is considered here. Namely, using the lines of $M=5$ and $M=11$ in Fig. 7, the $\omega/\lambda-k$ plane is divided into three regions as shown in Fig. 8. The first gives the almost overturning area. In the second, the column will overturn or not, but in practice, considering the real conditions as damping effect, it will hardly overturn. The third gives the non-overturning area.

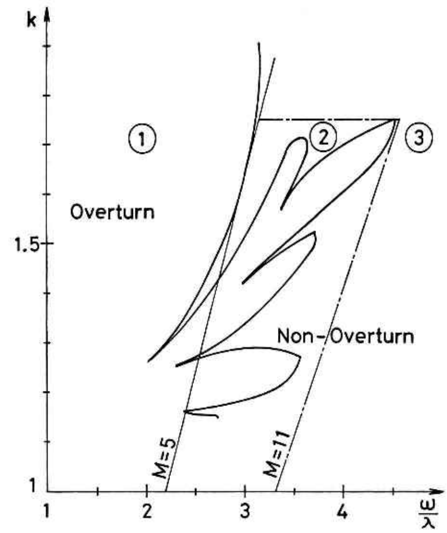


**Fig. 8** Overturning diagram for 3-wave excitation

Using the relations of $K$-$k$ and $\omega/\lambda$-$f$, the above three regions can be translated into those in $f$-$K$ plane. By such translations the overturning conditions of some

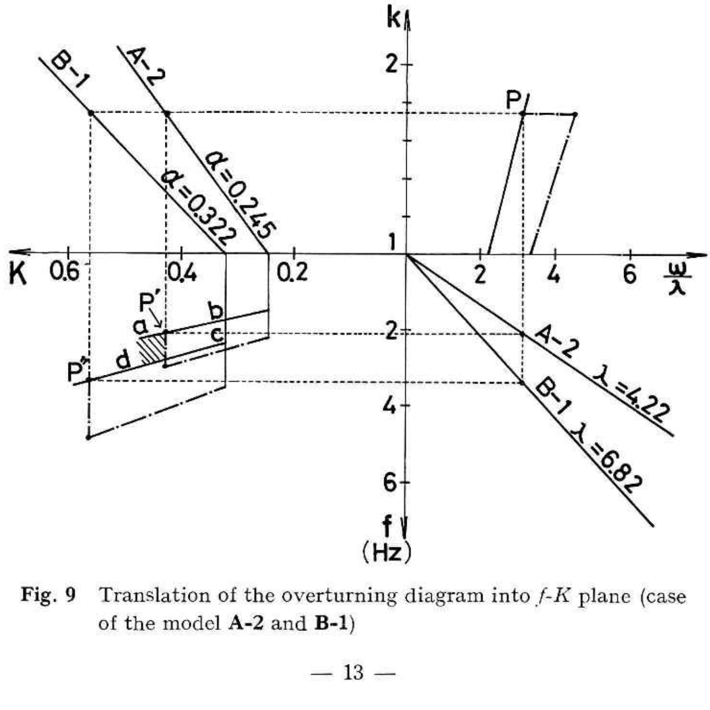


**Fig. 9** Translation of the overturning diagram into $f$-$K$ plane (case of the model **A-2** and **B-1**)

columns can be compared. As an example, the case of the models A-2 and B-1 are shown in Fig. 9. In the range of *abcd* in Fig. 9, the column B-1 will overturn, while the column A-2 will overturn or not, and especially it will not overturn in the shadowed area. Fig. 10 shows the responses of both models in the shadowed area. Like this the column, which has a smaller ratio of width to height and more possibility of overturning in appearance, does not always easily overturn comparing with the column which has a larger ratio and less possibility of overturning in appearance, if the former has a larger size than the latter.

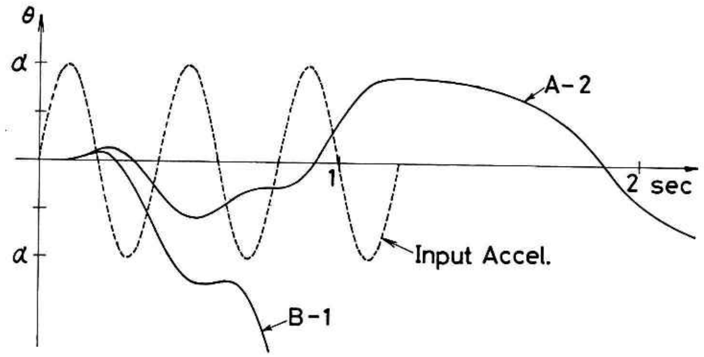


**Fig. 10** The responses of two models in the shadowed area in Fig. 9 (case of $f=2.5$ Hz and $K=0.44$)

On this phenomenon, some actual data and theoretical study from the point of view of energy were given by Ikegami and Kishinoue (1947, 1950) in the first place. In the present work it was found that the above phenomenon was generally able to occur in the case of the overturning caused by several waves excitation.

## 5. Conclusion

A consideration and numerical simulations on overturning of a simple rocking model have been done. The results obtained are as follows:

(1) The condition for overturning or stability of columns submitted to the sinusoidal excitation is represented by a region in $\omega/\lambda$-$k$ plane, and this region depends on the wave number of excitation and the damping ratio of angular velocity at collision.

(2) When the wave number of excitation and the damping ratio are constant, two similarity laws are derived from the above result. Namely, $S$ times the acceleration amplitude of excitation is nearly equivalent to $1/S$ times the ratio of width to height of a column, and $S$ times the frequency of excitation is equivalent to $S^2$ times the size of a column.

(3) By translating an overturning diagram in $\omega/\lambda$-$k$ plane into $f$-$K$ plane, the overturning of columns with various sizes can be compared. As an example, the case of 3-wave excitation and non-damping was shown.

(4) When the relation (32) holds, the column will overturn without collision after its first motion. A relatively low frequency and a large amplitude of excitation are required for this kind of overturning.

(5) When the relation (39) holds, the column will rock more than $M$ times. This condition gives a rough estimation for the safety of a column.

The author thinks that an experimental study is remaining to be done in this work and the effects of vertical excitation must be taken into account in future.

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# 角柱のロッキングと転倒に関する一考察

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角柱のロッキングと転倒については，墓石の転倒を利用した地震動の推定と関連して古くから調べられており，最近では地震時の家具，機器等の転倒防止の点からも関心がもたれている．本稿では従来から取り扱われてきた単純なロッキングモデルを考え，正弦波加振による転倒安定について相似律を含め二，三の特徴について検討した．ロッキング運動の際に生ずる衝突現象は重要な問題点であるが，本稿では簡単のために衝突時に一定の減衰率でエネルギーが失われるものと仮定した．

転倒条件については従来から静的加速度による判定法が使われており，またその欠点を補うためにエネルギー的観点にもとづく評価も行なわれてきたが，転倒に至るまでの動的な挙動については未知の点が多い．すなわち，静的加速度による判定は単に浮き上がるための必要条件を与えるにすぎないことはしばしば指摘されている通りであり，またエネルギーを考慮する方法は，1/4 正弦波加振を考え，地動のなした仕事と角柱のポテンシャルエネルギーを比較し，前者が大きい場合は転倒するものとしている．これらの方法では角柱がロッキング運動をくり返す場合についての考察ははじめから除外されている．

ロッキング運動を振動論的に取り扱う場合，衝突現象を本稿のように単純化して扱っても，衝突前後の振動方程式の切り換えによる非線形性が残ることになり，これが一般的な取り扱いを困難にしているものと思われる．

正弦波加振をうける角柱のロッキング運動では，最初の数波加振で転倒しない場合は周期解に漸近するものと思われ，この意味で初期の過渡的な応答が重要となる．本稿ではこれについて二つの点を明らかにした．一つは無限に続く正弦波加振をうける角柱が浮き上がりの後，再び床面に衝突することなく転倒に至る（従って衝突によるエネルギー減衰は考えなくてよい）ための条件を示し，その場合の最少加振波数についても検討した．低周波数，大振幅の地動による転倒はこの条件で与えられる．もう一つは安全側の評価についてである．すなわち，角柱が $M$ 回以上ロックするための一つの十分条件を示し，加振波数との関連について近似的な検討を行った．この条件は衝突時のエネルギー減衰がないものと仮定して得た．$M$ がある程度大きい場合，角柱のロッキングは周期解に近づき転倒しないことになる．これら二つの条件はいずれも $\omega/\lambda$（無次元振動数）を横軸，$k$（無次元振幅）を縦軸とする平面で転倒側および安全側の二つの領域を与える．この二つの領域の中間部分は $M$ が小さい，すなわち数回ロックして転倒に至る場合を含み，過渡応答の様相も複雑で取扱いが困難となる．この場合を含め，一般に正弦 $N$ 波加振による転倒安定を扱うには，一つのモデルによる数値シミュレーションの結果を相似律を介して種々の角柱に適用するしかないと思われる．これについて本稿では従来から知られている振幅および振動数に関する二つの相似律が成立つには加振波数および衝突時のエネルギー減衰率が一定という制限条件が必要であること，その場合には過渡的なロッキング運動からの転倒を含めてこれらの相似律が成り立つことを指摘した．この相似律は角柱の転倒安定が $\omega/\lambda-k$ 平面のある領域で与えられることと等価である．衝突によるエネルギー減衰率は一般に角柱の形状，衝突の反撥係数等に依存することが従来の研究で明らかになっている．したがってこれらの相似律は近似的にしか適用できない．特定地点の地震動を正弦波で代用する場合，加振波数は一定であり，またほぼ同程度の形状，基盤条件をもつ角柱の集合を考える場合，エネルギー減衰率はほぼ一定と考えられる．この意味で $\omega/\lambda-k$ 平面での転倒安定線図をあらかじめ数値シミュレーションによって求めておけば種々の大きさをもつ角柱の動的挙動について近似的な判定が行える．本稿ではその一例として 3 波加振の場合の結果を示した．またこの結果を用いて，形状からみて転倒しやすいものでもスケールが大きい場合は逆に転倒し難いという従来から知られている現象，さらにこれと対照的に相似な二つの角柱で大きいものが転倒し，小さいものが転倒しない場合も存在しうることを指摘した．