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A New Approach to the Heat Conduction Equation of Mixed Material and Its Application to Soil Systems

— Direct Measurement of Soil-Water Flow (1) —

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Abstract

The heat conduction equation of well-mixed materials is theoretically developed based on hypotheses which characterize mixed material. The developed equation is a modification of the heat conduction equation which was originally developed by the same writer for porous media consisting of two media : flowing medium and stationary medium. In the developed equation, there is no restriction on the number of components in the mixed material, and the components are considered to include stationary and flowing parts. Introducing the new definition of the components, application of the developed equation is widened. The heat conduction equations of soil systems in different situations are shown as examples of its application.

1. Preface

Thermal conductivity of soil has been studied theoretically (Tsao, 1961), (Shirai, *et al.*, 1977) and experimentally (Erh, *et al.*, 1971), (Sato, 1982), (Ishida, *et al.*, 1983). In general, considerations in the theoretical approach are based on the geometrical shape of each material of which the soil system is composed. Since the shape of each component is generally random, this approach causes difficulties in application of their developed thermal conductivity. On the other hand, the experimental approach is conducted to analyze what cannot be explained by the theoretical approach, though application of the results is restricted to stationary conduction in which the soil system does not change. From a practical point of view, the differential type of heat conduction equation is more useful than the thermal conductivity only for analyzing the dynamical characteristics of the objects.

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The writer has developed the heat conduction equation for prous media considering two types of components: stationary medium and flowing medium (Tominaga, 1976). This definition is simple, though there are difficulties in applying it, for the equation does not conveniently express cases where, for example, a part of liquid cannot move because of the friction of the surface of solid particles and only the remaining fluid part moves. This requires more suitable definitions for treating mixed material in which components are classified not only by the kind of materials but also by the existing situations: stationary and flowing. In the following sections, the heat conduction equation of mixed materials based on the two hypotheses characterizing mixed material are presented, and its applications to soil systems are shown.

Nomenclature

θ	K	temperature
u	K/s	temperature input
t	s	time
a	kg/m^3	density
c	J/(kg·K)	specific heat
λ	J/(m·s·K)	thermal conductivity
q	J/(m^3·s)	input energy
Q	J/m^3	internal energy
$\boldsymbol{v}$	m/s	velocity
$\boldsymbol{J}$	J/(m^2·s)	energy effluence density
$\boldsymbol{n}$		unit vector normal to the surface S*
n		number of the components
N_i		ratio of S_i to S, simultaneously V_i to V*
F_i		ratio of flowing area to total area of component i
$\bar{K}$	m^2/s	apparent thermal diffusivity
H		coefficient of apparent velocity
i		subscript for component
*S	m^2	area of closed surface S
S_i	m^2	area occupied by component i on closed surface S
V	m^3	volume of closed surface S
V_i	m^3	volume occupied by component i in S

2. Mixed material

When components of mixed material are not uniformly shaped, the boundary conditions initializing the equation cannot be set. Therefore, the following two hypotheses which are modifications of the originals (Tominaga, 1976) are introduced to characterize mixed material.

Hypothesis 1

On any closed surface S lying in mixed material, any N_i, which is the ratio of the area of sectional surface S_i occupied by a component i to the total area of the surface S, is uniform. where

$$N_i = \frac{S_i}{S}, \quad \sum_{i=1}^{n} N_i = 1 \tag{1}$$

The ratio N_i is the same as the ratio of volume of the component i to the total volume of the closed surface (Tominaga, 1980).

Hypothesis 2

In any infinitesimal portion, temperatures of the components are the same.

Necessity of **Hypothesis 2** is as follows: Since the components are well mixed on **Hypothesis 1**, the mixture is uniform, namely the conductivity is isotropic. This means that the temperature distribution is continuous in any finite portion. Therefore, the temperatures of the components should be the same in any infinitesimal portion.

3. Equation of heat conduction

3.1 Energy balance

The heat conduction equation is derived in a similar manner to Tominaga (1976). Input energy is equal to the sum of the increase of internal energy, the effluence of energy by heat conduction and the effluence of energy by flow in any closed surface in the mixed material. Supposing the volume is surrounded by the closed surface S, the energy balance is expressed as

$$\int_V q\mathrm{d}v=\int_V \frac{\partial Q}{\partial t}\mathrm{d}v+\int_S \boldsymbol{J}\cdot\boldsymbol{n}\mathrm{d}s+\int_S Q\boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}s \tag{2}$$

where the heat source is in the closed surface.

$\boldsymbol{v}$ is "real" velocity of liquid flow, but in mixed material it becomes "apparent" velocity explained in the section of **3.2 (C).**

3.2 Energy equation of mixed material

(A) Increase of internal energy

(The first term on the right-hand side of **Eq. (2)**)

Total energy $Q\mathrm{d}v$ of the volume $\mathrm{d}v$ is the sum of the energy of each component:

$$Q\mathrm{d}v=\sum_{i=1}^{n} Q_i\mathrm{d}v_i \tag{3}$$

where $Q_i\mathrm{d}v_i$ is the energy of the component i occupying $\mathrm{d}v_i$, and

$$Q_i=a_ic_i\theta_i \tag{4}$$

From **Hypothesis 1**

$$\mathrm{d}v_i=N_i\mathrm{d}v \tag{5}$$

From **Hypothesis 2**

$$\theta_i=\theta \tag{6}$$

The total energy is then written as

$$Q\mathrm{d}v=\sum_{i=1}^{n} a_ic_i\theta N_i\mathrm{d}v=\overline{ac}\,\theta\mathrm{d}v \tag{7}$$

where

$$\overline{ac}=\sum_{i=1}^{n} a_ic_iN_i \tag{8}$$

(B) Effluence of energy by heat conduction

(The second term on the right-hand side of **Eq. (2)**)

The flux of energy flow $\boldsymbol{J}\cdot\boldsymbol{n}\mathrm{d}s$ through the portion $\mathrm{d}s$ of the closed surface S is the sum of the flux of each component:

$$\boldsymbol{J}\cdot\boldsymbol{n}\mathrm{d}s=\sum_{i=1}^{n}\boldsymbol{J}_i\cdot\boldsymbol{n}\mathrm{d}s_i \tag{9}$$

where

$$\boldsymbol{J}_i = -\lambda_i \nabla \theta_i \tag{10}$$

By **Hypothesis 1**

$$\mathrm{d}s_i = N_i \mathrm{d}s \tag{11}$$

Eq. (9) becomes, from **Eqs. (10), (6)** and **(11)**

$$\begin{aligned} \boldsymbol{J} \cdot \boldsymbol{n} \mathrm{d}s &= \sum_{i=1}^{n} (-\lambda_i \nabla \theta) \cdot \boldsymbol{n} N_i \mathrm{d}s \\ &= -\overline{\lambda} \nabla \theta \cdot \boldsymbol{n} \mathrm{d}s \end{aligned} \tag{12}$$

where

$$\overline{\lambda} = \sum_{i=1}^{n} \lambda_i N_i \tag{13}$$

(C) Effluence of energy by flow

(The third term on the right-hand side of **Eq. (2)**)

For each component, there exists a part which flows and a part which does not flow. Introducing F_i which is the ratio of the flowing area to the total area for each component i on any closed surface S, the flowing part of a component through the closed surface S is

$$F_i S_i = F_i N_i S \tag{14}$$

where

$$0 \leq F_i \leq 1 \tag{15}$$

The real velocity of the flowing part in mixed material is not uniform. It varies not only with the location but also with the time. Thus, it is better to define the velocity as the rate of volume of flowing parts divided by the effective sectional area through which the components flow. Therefore, the total energy flowing out through the surface is the sum of the energy of the flowing parts of each component.

Then, from **Eq. (14)**

$$Q \boldsymbol{v} \cdot \boldsymbol{n} \mathrm{d}s = \sum_{i=1}^{n} Q_i \boldsymbol{v} \cdot \boldsymbol{n} F_i \mathrm{d}s_i \tag{16}$$

By **Eqs. (4), (6) and (11), Eq. (16)** becomes

$$Q\boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}s=\sum_{i=1}^{n}a_ic_i\theta\boldsymbol{v}\cdot\boldsymbol{n}F_iN_i\mathrm{d}s$$
$$=\overline{acF}\,\theta\boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}s \tag{17}$$

where

$$\overline{acF}=\sum_{i=1}^{n}a_ic_iF_iN_i \tag{18}$$

(D) Input energy

(The left-hand side of **Eq. (2)**)

The total energy which is added to the volume dv from point heat source placed in the volume v is the sum of the energy added to each component:

$$q\mathrm{d}v=\sum_{i=1}^{n}q_i\mathrm{d}v_i \tag{19}$$

The energy which is added to each component i is written by the temperature difference u_i between the point heat source and the component:

$$q_i=a_ic_iu_i \tag{20}$$

From **Hypothesis 2**

$$u_i=u \tag{21}$$

Then, from **Eqs. (20), (21), (5)** and **(8), Eq. (19)** becomes

$$q\mathrm{d}v=\sum_{i=1}^{n}a_ic_iuN_i\mathrm{d}v=\overline{ac}\,u\mathrm{d}v \tag{22}$$

(E) Heat conduction equation of mixed material

Substituting **Eqs. (7), (12), (17)** and **(22)** into **Eq. (2)**

$$\int_V\overline{ac}\,u\mathrm{d}v=\int_V\frac{\partial(\overline{ac}\theta)}{\partial t}\mathrm{d}v+\int_S(-\overline{\lambda}\,\nabla\theta)\cdot\boldsymbol{n}\mathrm{d}s+\int_S\overline{acF}\,\theta\boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}s \tag{23}$$

Applying Gauss' theorem to the second and third terms of the right-hand side of **Eq. (23),** and if $\overline{ac}$ does not change in the course of time at any spatial locations, **Eq. (23)** becomes

$$\frac{\partial\theta}{\partial t}=\nabla\cdot(\overline{K}\,\nabla\theta)-\nabla\cdot(H\theta\boldsymbol{v})+u \tag{24}$$

$\overline{K}$ is the apparent thermal diffusivity and H is the coefficient of apparent velocity.

where

$$\overline{K}=\frac{\overline{\lambda}}{\overline{ac}} \tag{25}$$

$$H=\frac{\overline{acF}}{\overline{ac}} \tag{26}$$

Eq. (24) is the general form of the heat conduction equation for mixed material including stationary and flowing parts of each component. Expression of **Eq. (24)** is similar to the equation presented in the paper of Tominaga (1976), though the expression of $\overline{K}$ and H are different from the former one, because the former equation was established on porous media composed of two media : flowing medium and stationary medium.

4. Heat conduction equation for soil

Eq. (24) is the general form for heat conduction of mixed material, so applications of the equation for soil are shown in this chapter.

Soil is classified into four situations : the water-saturated and water-unsaturated situations simultaneously the flow-existing and the flow-unexisting situations (see **Table 1**). In the following sections, the materials composing soil systems which are mineral particles, water and air are indicated by the subscripts of s, w and a.

4.1 Flow-unexisting situation

Since every component is stationary, the second term of the right-hand side of **Eq. (24)** vanishes :

$$\frac{\partial\theta}{\partial t}=\nabla\cdot(\overline{K}\nabla\theta)+u \tag{27}$$

(A) Saturated soil

There is no air in the soil system :

$$N_a=0$$

Then, the apparent thermal diffusivity becomes

$$\overline{K}=\frac{\lambda_S N_S+\lambda_w N_w}{a_S c_S N_S+a_w c_w N_w} \tag{28}$$

(B) Unsaturated soil

The apparent thermal diffusivity is

Table 1 Four situations of soil. The soil systems are classified by the water-saturated and the water-unsaturated situations simultaneously by the flow-unexisting and the flow-existing situations.

	Flow-unexisting $v=0$	Flow-existing $v\neq 0$
Saturated	$F_i=0$ $N_a=0$	$F_i\neq 0$ $N_a=0$
Unsaturated	$F_i=0$ $N_a\neq 0$	$F_i\neq 0$ $N_a\neq 0$

$$\overline{K}=\frac{\lambda_S N_S+\lambda_w N_w+\lambda_a N_a}{a_S c_S N_S+a_w c_w N_w+a_a c_a N_a} \tag{29}$$

4.2 Flow-existing situation

Since velocity does not vanish, the heat equation is the same as **Eq. (24)** :

$$\frac{\partial \theta}{\partial t}=\nabla \cdot(\overline{K}\nabla \theta)-\nabla \cdot(H\theta \boldsymbol{v})+u \tag{24}$$

(C) Saturated soil

$\overline{K}$ is the same as **Eq. (28)** :

$$\overline{K}=\frac{\lambda_S N_S+\lambda_w N_w}{a_S c_S N_S+a_w c_w N_w} \tag{28}$$

From **Eq. (26)**, the coefficient of apparent velocity H becomes

$$H=\frac{a_S c_S F_S N_S+a_w c_w F_w N_w}{a_S c_S N_S+a_w c_w N_w} \tag{30}$$

If the mineral particles do not move, and only the water part flows :

$$F_S=0$$

Hence

$$H=\frac{a_w c_w F_w N_w}{a_S c_S N_S+a_w c_w N_w} \tag{31}$$

$F_w N_w$ is the ratio of the effective area for water flow to the total area of any closed surface. .This value is generally difficult to measure.

(D) Unsaturated soil

The apparent thermal diffusivity $\bar{K}$ is the same as **Eq. (29)** :

$$\overline{K}=\frac{\lambda_S N_S+\lambda_w N_w+\lambda_a N_a}{a_S c_S N_S+a_w c_w N_w+a_a c_a N_a} \tag{29}$$

The coefficient of apparent velocity becomes

$$H=\frac{a_S c_S F_S N_S+a_w c_w F_w N_w+a_a c_a F_a N_a}{a_S c_S N_S+a_w c_w N_w+a_a c_a N_a} \tag{32}$$

If the mineral particles do not flow :

$$F_S=0$$

then

Table 2 Typical values of the density, specific heat and thermal conductivity of the components of soil system. The values are given under room temperature and normal pressure.

	Density a (kg/m^3)	Specific Heat c (J/(kg·K))	ac (J/(m^3·K))	Thermal Conductivity λ (J/(m·s·K))
Mineral Particle	2.5×10^3	8.4×10^2	2.1×10^6	8.4×10^{-1} -2.5×10^0
Water	1.0×10^3	4.2×10^3	4.2×10^6	5.9×10^{-1}
Air	1.2×10^0	1.0×10^3	1.2×10^3	2.4×10^{-2}

$$H=\frac{a_w c_w F_w N_w + a_a c_a F_a N_a}{a_S c_S N_S + a_w c_w N_w + a_a c_a N_a} \tag{33}$$

Between the values of density, specific heat, diffusivity of each component, there exist following relations (see **Table 2**) :

$$a_a c_a \ll a_S c_S,\ a_w c_w \tag{34}$$

$$\lambda_a \ll \lambda_S,\ \lambda_w \tag{35}$$

Therefore, the term of air can be ignored in **Eqs. (29)**, **(32)** and **(33).** Hence, **Eq. (24)** becomes finally

$$\frac{\partial\theta}{\partial t}=\nabla\cdot(\overline{K}_{SOIL}\nabla\theta)-\nabla\cdot(H_{SOIL}\theta\boldsymbol{v})+u \tag{36}$$

where

$$\overline{K}_{SOIL}=\frac{\lambda_S N_S+\lambda_w N_w}{a_S c_S N_S + a_w c_w N_w} \tag{37}$$

$$H_{SOIL}=\frac{a_w c_w F_w N_w}{a_S c_S N_S + a_w c_w N_w} \tag{38}$$

This is the heat conduction equation of commonly exsting soil.

5. Conclusion

The general form of the heat conduction equation for mixed material is developed theoretically based on the two hypotheses which characterize mixed material. Examples of the application of the obtained equation are shown for the heat conduction of soil. Since the hypotheses do not restrict the number of the components composing the mixed material, the result is widely applicable for any types of well-mixed material. As the heat conduction equation is presented in the form of differential equations, the equation can be applied to the analysis of dynamical characteristics of heat distribution of mixed material. In addition to this, since the apparent thermal diffusivity $\bar{K}$ and the coefficient of apparent velocity H appearing in the equation are defined explicity, the equation can be used for synthesis of mixed material, namely the heat conduction characteristics and estimates can be made even for materials being considered for the first time.

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混合物体の新しい熱伝導方程式と土壌系への適用
―土壌水の断面平均流量の直接測定(1)―

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多種類の物質からなる混合体の熱伝導方程式を2つの仮定にもとづき導出した．既発表の理論（富永，1976）では混合体を構成する物質として静止媒体と流動媒体の2種のみを考慮したが，本論文では一般の混合体の熱伝導現象を表現できるよう理論の拡張を行なったものである．構成物質の形状にとらわれない仮定に立ち熱伝導方程式の一般形を導出したので，従来の特定形状にもとづく解析や熱伝導率のみを対象とした研究にくらべ広い応用が可能である．応用の方法を示す例として土粒子，水，空気からなる土壌系をとりあげ，完全飽和および不飽和，かつ，流動ありおよび流動なしの4つの場合についての熱伝導方程式を示した．

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