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A Model of Turbulence: Statistical Treatment of an Ensemble of Vortex Filaments

By

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Abstract

Turbulence is modelized as a motion due to an ensemble of circular vortex filaments. By the local isotropic hypothesis of Kolmogoroff this ensemble can be thought of as a canonical ensemble of classical statistical mechanics. From this statistics, formulation is made for evaluating the velocity correlation function and energy spectrum tensors in isotropic turbulence of incompressible fluid.

1. Introduction

In most of practically important phenomena of fluid dynamics, various physical quantities such as velocity, pressure and temperature can not be determined uniquely, and fluctuating randomly in space and time, they are called turbulent. Turbulence has been one of the most difficult and most attractive problems in fluid dynamics and we have as yet no satisfactory theory of turbulence, though about forty years have elapsed sience L. Prandtl proposed his mixing-length theory. This theory can explain the universal logarithmic law near a wall and has been successfully used in various problems of turbulence due to its simplicity. These facts show that its physical picture is consistent with the real turbulent flow.

A new scope in turbulence theory has been opened when G. I. Taylor introduced the method of correlation function in 1935. This theory is a rather mathematical approach. We can obtain the equation of an arbitrary order velocity correlation tensor and the introduction of some assumption in regard to tensor of a certain order makes the problem closed and solved. E. Hopf's functional equation (E. Hopf, 1952; E. Hopf *et al.*, 1953) contains the information of correlation functions of all orders and might be said to be the widest generalization of correlation function method. Though a functional equation is hard to be solved, it may prove to be useful in the general discussion. Recently R. H. Kraichnan (1960) calculated the nonlinear interaction term by the perturbation method using graphical technique and succeeded in obtaining the $(-5/3)$ power law for isotropic turbulence in his latter paper (R. H. Kraichnan, 1965).

There have been many investigations of turbulence, but almost all of them are concerned with the homogeneous and isotropic turbulence, excepting W. V. R. Malkus (1956) who derived the mean velocity distribution of flow between the parallel plates using the maximum dissipation hypothesis. T. Imamura, W. C. Meecham and others (1963 and 1968) expanded the velocity field by Wiener-

Hermite functional, and from the Navier-Stokes equation they got equations satisfied with its expansion coefficients, and they successfully applied these to the problem of turbulence decay near a wall.

Thus, the recent trend of research in turbulence is strongly mathematical and tends to discard the physical entity. The present author thinks that a true solution of turbulence can not be attained without the understanding of physical picture of turbulent motion. In the present paper an ensemble of vortex filaments is adopted as a physical entity which causes turbulence, and is treated by the method of classical statistical mechanics.

2. Formulation

As is well-known in vector analysis, an arbitrary vector field can be expressed as a sum of irrotational and rotational vectors. In fluid mechanics the rotational part is usually the vorticity $\boldsymbol{\omega}$ which can be expressed by the velocity vector $\boldsymbol{u}$ as

$$\boldsymbol{\omega}=\nabla\times\boldsymbol{u} ,$$

where ∇ denotes the gradient operator.

Now, in case of incompressible fluid when the flow tends to zero sufficiently fast in infinite (*i.e.* $\boldsymbol{u}(\infty)=O(R^{-n})$, $n>3$), the velocity at any point in space can be expressed by the vorticity distribution as

$$u=\frac{\partial H}{\partial y}-\frac{\partial G}{\partial z}, \qquad v=\frac{\partial F}{\partial z}-\frac{\partial H}{\partial x}, \qquad w=\frac{\partial G}{\partial x}-\frac{\partial F}{\partial y},$$

where x, y, z are the orthogonal coordinates and $\boldsymbol{u}=(u, v, w)$. And

$$(F, G, H)=\frac{1}{4\pi}\int\frac{(\xi', \eta', \zeta')}{r}\mathrm{d}\tau' ,$$

where r is the distance between the point (x, y, z) and the point (x', y', z') at which volume element of integral $\mathrm{d}\tau'$ is situated and $\boldsymbol{\omega}=(\xi, \eta, \zeta)$ (Lamb, 1932). So we can determine the velocity field of incompressible fluid at least in principle if we know the distribution of vorticity in total space. But the introduction of assumption as to the distribution of vorticity at every point in space is as difficult as that of velocity distribution and an approach to the turbulence from the standpoint of vorticity has no merit, and only introducing complexity. So we do not enter into the vorticity distribution at any point in space, and we consider vortex filaments. Outside a vortex filament there exists velocity potential, and the potential caused by an ensemble of vortex filaments becomes the sum of individual potentials if they are not overlapping, and the problem is treated as a linear one. Of course, the fundamental feature of turbulence consists in its nonlinear interaction and this nonlinearity can not be excluded by any technique of linearization. In fact vortex filaments interact with each other and are born or dead at every instant and may be in a stationary state. But as the Lagrangian approach of Navier-Stokes equation gives the linear approximation higher by an order than that of Eulerian equation (Pierson, 1962), the linearization of turbulent field making use of vortex filaments is expected to give a better model than that from Eulerian description.

In the present paper, the interaction of vortex filaments is assumed to be so weak that it only exchanges energy and does not create a new filament nor destruct other vortex filaments. In another paper we may see the effect of creation and annihilation property of vortex field. So this approach may well be said to be a model of turbulence using a solid vortex filament.

As is well-known in hydrodynamics, a vortex filament neither begins nor ends at any point in space, but forms closed curve or has both ends at the surface of boundary. Of course, if we take account of the property of non-conservation of vortex, this assertion will not apply. And as we consider only the isotropic turbulence in this paper, all the vortex filaments in the region may well be assumed to be closed curve. Since the isotropic turbulence exists in the small region compared with the scale of field, larger vortex filaments with an end at the surface are thought to have an effect on that small region just like a mean flow.

Now we assume all the vortex filaments to be closed curves, but the closed curves have an infinite number of forms and for simplicity we must introduce some assumption as to the form of closed vortex filaments. First, we assume that all the vortex filaments are simple ones and do not entangle themselves. This statement is justified by the decomposability of entangled vortex filaments into a number of simple curves of filaments (Moffat, 1969). Secondly, we adopt for a closed curve its simplest form, a circle. This restriction may not give any serious modification and essential characteristics will be retained.

3. Statistics

As is described above, isotropic turbulence is thought to be an ensemble of a great number of circular vortex filaments. Behaviors of a system of many composites are treated by statistical mechanics. Consider a relatively small region of fluid moving at the velocity of mean flow, where the isotropy is a good approximation. In this region there are many vortex filaments which interact with each other, with the mean flow and with larger vortex filaments, and then there may be created a condition of statistical equilibrium of stationary state. These statements may be justified by the Kolmogoroff's local isotropy hypothesis and the interaction may well be thought to be so weak that we should only consider the exchange of energy of a system of vortex filaments in the small region concerned. Such a system is called in classical statistical mechanics a canonical ensemble, and the probability of the system in the energy range of T to $T+\mathrm{d}T$ is proportional to $e^{-\beta T}$, where β is a constant determined by the absolute temperature of the external field having a very large degree of freedom compared with the system concerned.

Now the energy T of an ensemble of vortex filaments is

$$T=\frac{\rho}{4\pi}\sum_{i\neq j}\Gamma_i\Gamma_j\iint\frac{\mathrm{d}\boldsymbol{s}_i\cdot\mathrm{d}\boldsymbol{s}_j}{r_{ij}}, \tag{1}$$

where ρ is the density of the fluid, Γ_i the strength of i-th vortex, and r_{ij} denotes the distance between the point of line integral element $\mathrm{d}\boldsymbol{s}_i$ of i-th vortex and that of $\mathrm{d}\boldsymbol{s}_j$ of j-th vortex. Using this formula, we can calculate any mean value as an expected value. The constant β is here unknown, but may depend only on the representative quantity of the flow and can be determined by the comparison with experimental results. Phase space is composed of the vector in space coordi-

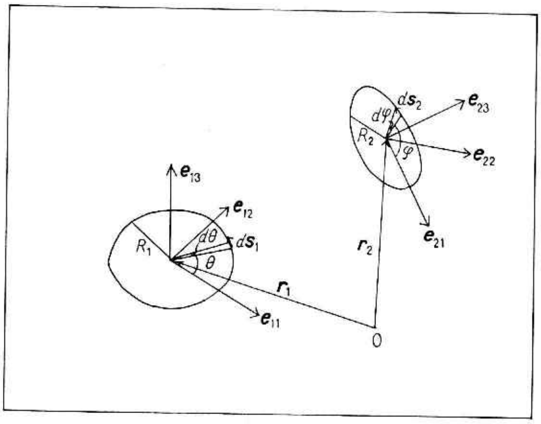

Fig. 1. Orthogonal coordinates of two vortex filaments.

nates $\boldsymbol{r}_i$ of the center of vortex filament, the unit vector $\boldsymbol{e}_{i3}$ normal to the plane of vortex, the radius of circle R_i and the strength Γ_i of vortex filament. Now we address ourselves to the problem of evaluation of energy of a system and velocity potential induced by them and the velocity correlation function.

4. Calculation

We rewrite the equation (1) as

$$T=\frac{\rho}{4\pi}\sum_{i\neq j}\Gamma_i\Gamma_j\Gamma_{ij}\ ,$$

where

$$T_{ij}=\iint\frac{\mathrm{d}\boldsymbol{s}_i\cdot\mathrm{d}\boldsymbol{s}_j}{r_{ij}}\ .$$

We calculate T_{12}, since from this form other T_{ij} can be easily inferred. We take $\boldsymbol{e}_{i1}$ and $\boldsymbol{e}_{i2}$ on the surface normal to $\boldsymbol{e}_{i3}$ such that three unit vectors $\boldsymbol{e}_{i1}$, $\boldsymbol{e}_{i2}$, $\boldsymbol{e}_{i3}$ construct an orthogonal system, and take $\boldsymbol{e}_{01}$, $\boldsymbol{e}_{02}$, $\boldsymbol{e}_{03}$ as orthogonal vectors fixed in the space. We can write

$$\boldsymbol{e}_{1i}=\sum_{j=1}^{3}p_{ij}\boldsymbol{e}_{0j}\ ,\qquad \boldsymbol{e}_{2i}=\sum_{j=1}^{3}q_{ij}\boldsymbol{e}_{0j}\ ,$$

where p_{ij} and q_{ij} are direction cosines between $\boldsymbol{e}_{1i}$ and $\boldsymbol{e}_{0j}$, and between $\boldsymbol{e}_{2i}$ and $\boldsymbol{e}_{0j}$, respectively. Line element $\mathrm{d}\boldsymbol{s}_1$ can be written in the $(\boldsymbol{e}_{11}, \boldsymbol{e}_{12}, \boldsymbol{e}_{13})$ system as

$$\mathrm{d}\boldsymbol{s}_1=(-R_1\sin\theta\,\mathrm{d}\theta,\ R_1\cos\theta\,\mathrm{d}\theta,\ 0)\ .$$

Using a transformation formula in a vector analysis

$$x_i=\alpha_i+\sum_{j=1}^{3}p_{ji}x_j'\ ,$$

where $\boldsymbol{\alpha}=(\alpha_1, \alpha_2, \alpha_3)$ denotes the position vector of an origin of a new coordinate system and in this case $\boldsymbol{\alpha}=\boldsymbol{r}_i$, we get

$$\mathrm{d}\boldsymbol{s}_1\cdot\mathrm{d}\boldsymbol{s}_2=R_1R_2\,\mathrm{d}\theta\,\mathrm{d}\varphi(A_{11}\sin\theta\sin\varphi-A_{12}\sin\theta\cos\varphi$$
$$-A_{21}\cos\theta\sin\varphi+A_{22}\cos\theta\cos\varphi)\ ,$$

where φ denotes the angle between $\boldsymbol{e}_{21}$ and the position vector of line integral element $\mathrm{d}\boldsymbol{s}_2$, and A_{ij} the inner product of vector $\boldsymbol{e}_{1i}$ and $\boldsymbol{e}_{2j}$, *i.e.*

$$A_{ij}=\boldsymbol{e}_{1i}\cdot\boldsymbol{e}_{2j}\ .$$

If we denote

$$\boldsymbol{r}_1-\boldsymbol{r}_2=\boldsymbol{r}_{12}\ , \tag{2}$$

we get

$$r_{12}=|\boldsymbol{r}_{12}|^2+R_1^2+R_2^2+2R_1(C_{11}\cos\theta+C_{12}\sin\theta)-2R_2(C_{21}\cos\varphi+C_{22}\sin\varphi)$$
$$-2R_1R_2(A_{11}\cos\theta\cos\varphi+A_{12}\cos\theta\sin\varphi+A_{21}\sin\theta\cos\varphi+A_{22}\sin\theta\sin\varphi)\ ,$$

where

$$C_{ij}=\boldsymbol{r}_{12}\cdot\boldsymbol{e}_{ij}\ .$$

If we assume that two votexes are enough apart so as to be

$$\left(\frac{R_i}{|\boldsymbol{r}_{12}|}\right)^3\fallingdotseq 0\ , \tag{3}$$

we can evaluate the line integral to such an approximation as

$$T_{12}=(R_1R_2)^2\frac{1}{|\boldsymbol{r}_{12}|^3}\Big[2(A_{11}A_{22}-A_{12}A_{21})-\frac{3}{|\boldsymbol{r}_{12}|^3}\{C_{11}(C_{21}A_{22}-C_{22}A_{21})$$
$$-C_{12}(C_{21}A_{12}-C_{22}A_{11})\}\Big]\ .$$

Now using the formula of vector analysis, $(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D})=(\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{B}\cdot\boldsymbol{D})-(\boldsymbol{A}\cdot\boldsymbol{D})(\boldsymbol{B}\cdot\boldsymbol{C})$, where $\boldsymbol{A}\times\boldsymbol{B}$ denotes the vector product of two vectors $\boldsymbol{A}$, $\boldsymbol{B}$, we get

$$A_{11}A_{22}-A_{12}A_{21}=\boldsymbol{e}_{13}\cdot\boldsymbol{e}_{23}$$

and

$$C_{11}(C_{21}A_{22}-C_{22}A_{21})-C_{12}(C_{21}A_{12}-C_{22}A_{11})=-(\boldsymbol{e}_{13}\cdot\boldsymbol{r}_{12})(\boldsymbol{e}_{23}\cdot\boldsymbol{r}_{12})+|\boldsymbol{r}_{12}|^2(\boldsymbol{e}_{13}\cdot\boldsymbol{e}_{23})\ ,$$

and finally we get the T_{12} as

$$T_{12}=(R_1R_2)^2\frac{1}{|\boldsymbol{r}_{12}|^3}\left\{-(\boldsymbol{e}_3\cdot\boldsymbol{e}_3')+\frac{3}{|\boldsymbol{r}_{12}|^2}(\boldsymbol{r}_{12}\cdot\boldsymbol{e}_{13})(\boldsymbol{r}_{12}\cdot\boldsymbol{e}_{23})\right\}\ .$$

Of course, this form does not depend on the arbitrary quantities $\boldsymbol{e}_{11}$, $\boldsymbol{e}_{12}$, $\boldsymbol{e}_{21}$ and $\boldsymbol{e}_{22}$, but depends on the strength of vorticity and the relative separation and mutual direction. Thus the total energy of a system of vortex filaments is

$$T=\frac{\rho\pi}{2}\sum_{i<j}\Gamma_i\Gamma_j(R_iR_j)^2\frac{1}{|\boldsymbol{r}_{ij}|^3}\left\{-(\boldsymbol{e}_{i3}\cdot\boldsymbol{e}_{j3})+\frac{3}{|\boldsymbol{r}_{ij}|^3}(\boldsymbol{r}_{ij}\cdot\boldsymbol{e}_{i3})(\boldsymbol{r}_{ij}\cdot\boldsymbol{e}_{j3})\right\}\ . \tag{4}$$

Next, we calculate the velocity potential due to an ensemble of vortex filaments. Outside a vortex filament the velocity field can be described by the velocity potential $\phi(\boldsymbol{r})$ and be written

$$\phi(\boldsymbol{r})=\frac{\Gamma_i}{4\pi}\int\frac{\cos\vartheta}{r^2}\,dS'\ ,$$

where ϑ denotes the angle between the vector $\boldsymbol{r}'-\boldsymbol{r}$ and the surface normal, and $\boldsymbol{r}'$ the position vector of surface integral element dS' (Lamb, 1932, §150). As velocity potential is a linear quantity, we can calculate the velocity potential due to an ensemble as the sum of individual potential ϕ_i induced by individual vortex filament,

$$\phi=\sum\phi_i\ .$$

Just as in the previous calculation for energy, we can calculate the velocity potential due to a circular vortex filament of strength Γ_i in the approximation

$$\left(\frac{R_i}{|\boldsymbol{r}_i'|}\right)^3\fallingdotseq 0 \tag{5}$$

as

$$\phi_i(\boldsymbol{r})=\frac{\Gamma_i}{4}R_i^2\frac{(\boldsymbol{r}_i'\cdot\boldsymbol{e}_{i3})}{|\boldsymbol{r}_i'|^3}\ ,$$

where

$$\boldsymbol{r}_i'=\boldsymbol{r}_i-\boldsymbol{r}\ . \tag{6}$$

Now we can compute velocity correlation function in the same manner as the evaluation of expected value in statistical mechanics. Velocity correlation function for two points $R_{ij}(\boldsymbol{r}-\boldsymbol{r}')=\overline{u_i(\boldsymbol{r})u_j(\boldsymbol{r}')}$ can be calculated as follows, where the overlaid bar denotes the expected value.

$$\begin{aligned} R_{ij}(\boldsymbol{r}-\boldsymbol{r}') &= \overline{\frac{\partial\phi(\boldsymbol{r})}{\partial x_i}\frac{\partial\phi(\boldsymbol{r}')}{\partial x_j'}} \\ &= \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j'}\overline{\phi(\boldsymbol{r})\phi(\boldsymbol{r}')}\ , \end{aligned}$$

and

$$\begin{aligned} \overline{\phi(\boldsymbol{r})\phi(\boldsymbol{r}')} &= \sum_{\alpha,\beta}\overline{\phi_\alpha(\boldsymbol{r})\phi_\beta(\boldsymbol{r}')} \\ &= \sum_{\alpha}\overline{\phi_\alpha(\boldsymbol{r})\phi_\alpha(\boldsymbol{r}')}+\sum_{\alpha\neq\beta}\overline{\phi_\alpha(\boldsymbol{r})\phi_\beta(\boldsymbol{r}')} \\ &= N\overline{\phi_1(\boldsymbol{r})\phi_1(\boldsymbol{r}')}+N(N-1)\overline{\phi_1(\boldsymbol{r})\phi_2(\boldsymbol{r}')}\ , \end{aligned}$$

where N is the number of vortex filaments. Evaluation of this integral and the discussion will be given in the next paper.

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乱流の一つのモデル： うず糸の集合の統計的処理

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乱流を数多くの円形のうず糸の集まりとみなす．局所等方性の仮定から非圧縮性流体の等方性乱流においてこのうず糸の集合が標準集団として扱えることを用いて，乱流場の速度相関，エネルギースペクトルテンソルを算出する定式化を行なった．