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Measurement of Water Movement through Soil

—Water Velocity Measurement using Sinusoidal Heat Input—

By

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Abstract

Mathematical development of the water velocity measurement using sinusoidal heat input is described. The principle of the method is to use the relation between the responses of temperatures obtained from two measuring points located on a line passing through the point heat source. The solution, which is a sinusoidal heat output responsive to the sinusoidal heat input, is derived from the energy equation in the fluid using Laplace transform method.

Two methods are obtained from this solution for measurement of the velocity of fluid. One is the method using the ratios of the amplitude of outputs and the other is the method using the phase differences between outputs. The merit and demerit of these two methods for practical use are mentioned.

1) Preface

Natural disasters such as slope failure and landslide have occurred during or shortly after heavy rainfall. Especially, the rainfall for two hours before the failure or slide has been supposed to play an important role. As slope failure and landslide are dynamical phenomena, not only the accumulation of rainfall but also the movement of pore water must be measured in order to make clear the causality of the phenomena. The velocity of water in soil is very slow as is obtained from Darcy's law or others, hence in the present paper some considerations should be given to the measurement of water velocity in soil. As fundamental part of a research project for measuring the water movement through soil, a method suitable for measurement of low velocity in fluid was developed. Sinusoidal heat signal was adopted as the input.

A number of phenomena concerning the relation between temperature and flowing fluid have been used for measureing the flow velocity, and they are classified into the following three types:

a) measurement of the heat transfer from a heat source to fluid,
b) measurement of the quantitative changes of temperature,
c) measurement of the pattern of temperature changes.

An example of a) is a hot-wire anemometer, which successfully utilizes the resistance changes in the hot wire caused by heat dissipation from the hot wire to the flowing fluid. The amount of heat dissipation is so easily influenced by physical circumstances that calibration is required for assurance of accuracy in each case of measurement. Methods of b) are designed so as to find velocities by calculating the changes of temperature. As the diffusivity of fluid is variable with physical circumstances, applications of b) have been hardly developed. The method measuring the response to a pulsive heat input and the method using the correlation function of the fluctuating temperature (Kashiwagi, 1971) are examples of c).

In the case that the conduction of heat need not be taken into account, the velocity can be measured accurately by the correlation function. But when the velocity is low, that is, when the conduction of heat cannot be ignored, variations of the pattern of temperature must be considered. The temperature output responsive to pulsive heat input is a relatively simple signal to be detected. Even in this method, error may increase when the conduction of heat cannot be ignored, because the analytical response to pulsive heat input, width of which is finite, is too complex to be used. The above discussion shows some difficulty in the fluid velocity measurement using the pattern of temperature.

R.E. Walker developed a method in which the phase shift of temperature pattern responsive to the sinusoidal heat input was measured for obtaining the velocity of gas (Walker, 1956). He used a hot-wire anemometer, because the high frequency characteristics were necessary in order to get high resolution of velocity. The phase shift varied with the velocity of gas, the frequency of input and the distance between two probes. One of the probes was a heat source, and the other a detector. An integral multiple (N) frequency, which was synchronized with the frequency of heat input, was applied to the abscissa of the oscilloscope, and the output voltage of the detector was applied to the ordinate. Hence the Lissajous pattern repeated an identical pattern 2 N times, while the detector was moved downstream until the phase of low frequency signal, which was applied to the ordinate, changed by 360 degrees. Ignoring the effect of heat conduction, Walker obtained the velocity by means of calculation of three factors, namely, the number of repetitions, the distance of detector movement and the frequency of input signal. If this method is applied to a flowing liquid, the discussion becomes more complicated, because the termal diffusivity of a fluid is not negligible generally.

In the present paper, the methods using sinusoidal heat input is discussed in order to measure the velocity of fluid. In the following sections, the analytical solution of the energy equation, in which the sinusoidal heat signal is applied as the input, will be derived. Two applications of this solution will be described. One is a method using the ratio of amplitudes of temperature outputs, and the other a method using the phase differences between the outputs of two serial measuring points.

2) Energy equation of fluid

In a portion bounded by a closed surface S in a fluid the input energy is equal to the sum of the increase of internal energy, the energy effluence by conduction and that by flow of liquid. This relation is written as

$$\int_V q \mathrm{d}v = \int_V \frac{\partial Q}{\partial t} \mathrm{d}v + \int_S \boldsymbol{J} \cdot \boldsymbol{n} \mathrm{d}s + \int_S Q \boldsymbol{v} \cdot \boldsymbol{n} \mathrm{d}s \tag{1}$$

where q is the input energy ($\mathrm{J \cdot m^{-3} \cdot s^{-1}}$), Q the internal energy ($\mathrm{J \cdot m^{-3}}$), $\boldsymbol{J}$ the vector of energy effluence density ($\mathrm{J \cdot m^{-2} \cdot s^{-1}}$), $\boldsymbol{n}$ the unit vector normal to the surface S. $\boldsymbol{v}$ the velocity of fluid, V the volume, and S area of the portion mentioned above. The second and third terms on the right side of Eq. (1) are changed into the integrals with volume by Gauss' theorem. If the density and specific heat of the fluid are uniform, Eq. (1) becomes

$$\frac{\partial \theta}{\partial t} = \nabla \cdot (K \nabla \theta) - \nabla \cdot (\theta \boldsymbol{v}) + f \tag{2}$$

where θ is the temperature (deg) of fluid, K the thermal diffusivity ($\mathrm{m^2 \cdot s^{-1}}$), f the input heat ($\mathrm{deg \cdot s^{-1}}$), and

$$\nabla = \boldsymbol{i} \frac{\partial}{\partial x} + \boldsymbol{j} \frac{\partial}{\partial y} + \boldsymbol{k} \frac{\partial}{\partial z} \tag{3}$$

$$\boldsymbol{v} = \boldsymbol{i} v_X + \boldsymbol{j} v_Y + \boldsymbol{k} v_Z \tag{4}$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are unit direction vectors, and v_X, v_Y, v_Z the components of $\boldsymbol{v}$ along x, y, z coordinates, respectively. If the unit impulsive heat signal is applied to a point of $\boldsymbol{L}$ and at a time of t', the response is

$$G(\boldsymbol{r}-\boldsymbol{L}, t-t') = \begin{cases} 0 & t < t' \\ \left(\dfrac{1}{2\sqrt{\pi K (t-t')}} \right)^3 \mathrm{e}^{-\frac{\|\boldsymbol{r}-\boldsymbol{L}-\boldsymbol{v}(t-t')\|^2}{4K(t-t')}}, & t' \leq t \end{cases} \tag{5}$$

where $\boldsymbol{r}$ is a position vector defined as

$$\boldsymbol{r} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$$

Eq. (5) is known as the Green function of heat conduction.

3) Response to a sinusoidal heat input

Supposing the thermal diffusivity to be uniform and the velocity of flow to be constant in a fluid, the distribution of temperature is determined by the heat input and the initial distribution of temperature. The following boundary conditions are proposed:

a) The initial distribution of temperature φ is

$$\lim_{\rho \to -\infty} \varphi(\boldsymbol{L}, \rho) = 0, \qquad 0 \leq \|\boldsymbol{L}\| \tag{6}$$

where $\varphi(\boldsymbol{L}, \rho)$ is the temperature at $\boldsymbol{r}=\boldsymbol{L}$ and $t=\rho$. In this expression, the constant component of temperature distribution is excluded.

b) The heat input f is

$$f=\cos wt, \quad \text{at } \boldsymbol{r}=\boldsymbol{o} \text{ and } t>-\infty \tag{7}$$

where $\boldsymbol{o}$ is the origin of the coordinates. The response, namely the distribution of temperature at $\boldsymbol{r}$ and t, is expressed as

$$\theta(\boldsymbol{r}, t)=\int_{-\infty}^{t} G(\boldsymbol{r}, t-\tau)\cos w\tau \mathrm{d}\tau+\lim_{\rho\to-\infty}\int_{-\infty}^{\infty} G(\boldsymbol{r}-\boldsymbol{L}, t-\rho)\varphi(\boldsymbol{L}, \rho)\mathrm{d}\boldsymbol{L} \tag{8}$$

The origin of time can be chosen arbitrarily. As $\cos w\tau$ is the real part of $e^{iw\tau}$, Eq. (8) is rewritten as

$$\theta(\boldsymbol{r}, t)=R_e\left[\int_{-\infty}^{t} G(\boldsymbol{r}, t-\tau)\mathrm{e}^{iw\tau}\mathrm{d}\tau\right] \tag{9}$$

where R_e denotes the real part of inner term in brackets. Transforming the variable τ to ξ by the relation of $t-\tau=\xi$, Eq. (9) becomes

$$\theta(\boldsymbol{r}, t)=R_e\left[\int_0^{\infty}\left(\frac{1}{2\sqrt{\pi K\xi}}\right)^3 \mathrm{e}^{-\frac{\|\boldsymbol{r}-\boldsymbol{v}\xi\|^2}{4K\xi}}\mathrm{e}^{iw(t-\xi)}\mathrm{d}\xi\right] \tag{10}$$

The exponential part of Eq. (10) is separated into two terms. One of them includes the variable ξ, but the other does not. Hence

$$\theta(\boldsymbol{r}, t)=R_e\left[\left(\frac{1}{2\sqrt{\pi K}}\right)^3 \mathrm{e}^{\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{2K}+iwt}\int_0^{\infty}\frac{1}{\xi\sqrt{\xi}}\mathrm{e}^{-\left(\frac{\|\boldsymbol{v}\|^2}{4K}+iw\right)\xi-\frac{\|\boldsymbol{r}\|^2}{4K\xi}}\mathrm{d}\xi\right] \tag{11}$$

The integral part of Eq. (11) is one of the typical forms of Laplace transform:

$$\int_0^{\infty}\mathrm{e}^{-p\xi}\frac{1}{\xi\sqrt{\xi}}\mathrm{e}^{-\frac{a^2}{4\xi}}\mathrm{d}\xi=\frac{2\sqrt{\pi}}{|a|}\mathrm{e}^{-|a|\sqrt{p}} \tag{12}$$

where a is a non-zero real number, p a complex number, real part of which is positive, and both of a and p being independent of ξ. Then Eq. (11) changes into

$$\theta(\boldsymbol{r}, t)=R_e\left[\frac{1}{4\pi K\|\boldsymbol{r}\|}\mathrm{e}^{\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{2K}+iwt\pm\frac{\|\boldsymbol{r}\|\|\boldsymbol{v}\|}{2K}(\alpha+i\beta)}\right] \tag{13}$$

where

$$\alpha=\sqrt{\frac{\sqrt{1+k^2}+1}{2}}, \qquad \beta=\sqrt{\frac{\sqrt{1+k^2}-1}{2}} \tag{14}$$

$$k=\frac{4Kw}{\|\boldsymbol{v}\|^2} \tag{15}$$

From physical consideration, the amplitude of temperature should tend to be zero when the distance of the point $\boldsymbol{r}$ from the origin becomes infinitely larger. Therefore, the upper part of the double sign in Eq. (13) is adopted.

Finally

$$\theta(\boldsymbol{r}, t)=\frac{1}{4\pi K\|\boldsymbol{r}\|}\mathrm{e}^{\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{2K}-\frac{\|\boldsymbol{r}\|\|\boldsymbol{v}\|}{2K}\alpha}\cos\left(wt-\frac{\|\boldsymbol{r}\|}{2K}\sqrt{\frac{\sqrt{\|\boldsymbol{v}\|^4+16K^2w^2}-\|\boldsymbol{v}\|^2}{2}}\right) \tag{16}$$

4) Velocity measurement

Eq. (16) gives two methods of measuring the velocity. One is the method using changes of the amplitude of temperature, and the other the method using phase differences.

a) Amplitude method

Let the point heat source be fixed at the origin of the coordinates, and let six measuring points be settled symmetrically on the coordinates like the vertexes of a regular octahedron. The outputs on the coordinate of x are expressed as

$$\theta_{\pm X} = \frac{1}{4\pi KR} e^{\frac{R}{2K}(\pm v_X - \|\boldsymbol{v}\|\alpha)} \cos\left(wt - \frac{R\|\boldsymbol{v}\|}{2K}\beta\right) \tag{17}$$

where θ_{+X} and θ_{-X} denote the outputs at the points of $\boldsymbol{r}=\boldsymbol{i}R$ and $\boldsymbol{r}=-\boldsymbol{i}R$ on the coordinate of x, and include $+v_X$ and $-v_X$ respectively. Fig. 1 shows the locations of these points.

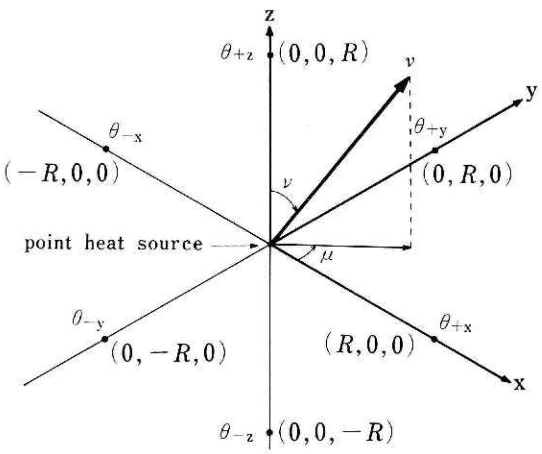

Fig. 1. Locations of measuring points.

The orthogonal coordinates system is set in fluid. Three pairs of the points are located on the three coordinates respectively. The distances between these points and the origin are the same and denoted with R. Heat input is applied at the origin. $\boldsymbol{v}$ is the vector of velocity of the fluid, of which the direction is expressed by the components of the circular cylindrical coordinates.

Using Eq. (17), the ratio of θ_{+X} to θ_{-X} are written as

$$\frac{\theta_{+X}}{\theta_{-X}} = e^{\frac{Rv_X}{K}} \tag{18}$$

The ratios of θ_{+Y} to θ_{-Y} and θ_{+Z} to θ_{-Z} are written by the expressions mentioned above. Next A, B, and C are defined as logarithms of these values on the circular cylindrical coordinates (see Fig. 1), hence

$$\left.\begin{aligned} A &= \ln\frac{\theta_{+X}}{\theta_{-X}} = \frac{Rv_X}{K} = \frac{R}{K}\|\boldsymbol{v}\|\sin\nu\cos\mu \\ B &= \ln\frac{\theta_{+Y}}{\theta_{-Y}} = \frac{Rv_Y}{K} = \frac{R}{K}\|\boldsymbol{v}\|\sin\nu\sin\mu, \\ C &= \ln\frac{\theta_{+Z}}{\theta_{-Z}} = \frac{Rv_Z}{K} = \frac{R}{K}\|\boldsymbol{v}\|\cos\nu \end{aligned}\right\} \tag{19}$$

The components of velocity are:

$$\left.\begin{aligned} \|\boldsymbol{v}\| &= \frac{K}{R}\sqrt{A^2+B^2+C^2} \\ \mu &= \mathrm{Tan}^{-1}\left(\frac{B}{A}\right) \\ \nu &= \mathrm{Tan}^{-1}\left(\frac{\sqrt{A^2+B^2}}{C}\right) \end{aligned}\right\} \tag{20}$$

or

$$\boldsymbol{v} = \frac{K}{R}(\boldsymbol{i}A+\boldsymbol{j}B+\boldsymbol{k}C) \tag{21}$$

The thermal diffusivity is indispensable for calculation of the velocity by this method.

b) Phase difference method

The phase difference between the simultaneous outputs of two measuring points, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, is expressed as

$$\phi = (\|\boldsymbol{r}_1\|-\|\boldsymbol{r}_2\|)\frac{\|\boldsymbol{v}\|}{2K}\beta \tag{22}$$

where the three points, namely, the heat source, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, stand on a straight line. ϕ is equal to the phase difference calculated by the travel time of a peak signal from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$. Then $\boldsymbol{v}$ is written as

$$\|\boldsymbol{v}\|^2 = \frac{(\Delta r)^2}{\phi^2}w^2 - 4K^2\frac{\phi^2}{(\Delta r)^2} \tag{23}$$

where

$$\Delta r = \|\boldsymbol{r}_2\| - \|\boldsymbol{r}_1\|, \quad 0 < \|\boldsymbol{r}_1\| < \|\boldsymbol{r}_2\| \tag{24}$$

When the value of K is known, $\|\boldsymbol{v}\|$ is immediately obtained. But in many cases the diffusivity is so changeable that it is impossible to assume K as a known value before the run of the experiment. If the two phase differences, ϕ_1 and ϕ_2, are respectively known in the responces to the two input angular frequencies, ω_1 and ω_2, in regard to one velocity, then it is possible to eliminate the diffusivity. Therefore,

$$\|\boldsymbol{v}\|^2 = (\Delta r)^2\left(\frac{w_1^2}{\phi_1^2}+\frac{w_2^2}{\phi_2^2}-\frac{w_1^2-w_2^2}{\phi_1^2-\phi_2^2}\right) \tag{25}$$

$$K^2 = \frac{(\Delta r)^4}{4}\frac{1}{\phi_1^2-\phi_2^2}\left(\frac{w_1^2}{\phi_1^2}-\frac{w_2^2}{\phi_2^2}\right) \tag{26}$$

The direction of flow cannot be obtained by the phase difference method, because phases at any points located at the same distance from the point source are the same. In other words, the phase difference is not affected by the direction of the flow.

5) Conclusion and Remarks

The sinusoidal heat input method has following merits for measuring fluid velocities:

a) Heat signal used for an input does not contaminate the fluid.

b) Continuous measurement is available.

c) An analytical solution supports a mathematical base.

d) The wide band amplifier is not required, because the output frequency which must be observed is perfectly the same as that of the input.

The phase difference method has additional merits as follows:

e) The conductivity of fluid is not required in the case that the responses to the two inputs of different frequencies are known, for frequency of a sinusoidal heat input is an independent variable. This means that the absolute velocity of flow can be measured without the conductivity.

f) As the phase differences are not affected by the amplitude of the output signals, the variation of impedance of the signal paths can be ignored.

Though the methods discussed in the present paper have several merits mentioned above, attention must be paid to the following remarks in practical use:

a) The energy equation of heat conduction is based on the hypothesis that energy is transferred from a warm point to a cold point without delay. This means that if heat source is applied to a point, the temperature change must be instantaneously detected at any point located far from the source. This is strange, because it takes a certain amount of time to transfer energy from a point to another. As the angle between the direction of the flow and that of heat conduction at a certain point is not always the same as that at an opposite point in respect to the heat source, the energy will not be transferred simultaneously to these points. Therefore, the phase of temperature response at one of these points may be different from that at the other point.

b) As the energy flow is proportional to the gradient of temperature, the energy input is not sinusoidal in the case that the temperature of fluid varies at heat source, even if the heat source gives sinusoidal temperature to the flow. When the frequency of input is sufficiently high and the amplitude is sufficiently small, the sinusoidal temperature input approximately gives the sinusoidal energy input. In such cases the temperature outputs are regarded as the small signal heat reponses superposed on the average temperatures.

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土中の水分移動量の測定
—正弦波状温度入力を利用した流速測定—

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正弦波状温度入力を利用する流速測定法が述べられている．この測定法は点熱源を通る直線上に位置した2つの測定点から得られる温度応答の相互関係を利用するものである．正弦波状温度入力に対する応答はラプラス変換を利用し熱伝導方程式を解くことにより数学的に求められている．上記の解を利用して2つの流速測定法が導かれている．一つは2点の温度応答の振幅の比を利用するものであり，他は位相差を利用するものである．これら二つの測定法を使用するに際しての長所と短所が述べられている．

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