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Measurement of Water Movement through Soil

——Mathematical Development of Heat Conduction in Porous Media——

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Abstract

A model of heat conduction through porous media is developed under two hypotheses. Two media are taken into account for the study of the transfer of heat energy. One is flowing medium, and the other stationary medium. The hypotheses are: I) volume porosity and surface porosity are constant everywhere in porous media, and II) the temperature of flowing medium and that of stationary medium are the same at any infinitesimal portions. The result has come to be similar in the models of one and the same uniform substance. As the idea of the model is universal, it will be widely applicable.

1) Preface

Heat conduction through saturated or unsaturated soil is complicated. Three substances, namely, mineral particles, water and air, take part in heat conduction. It is difficult to know the rate of contribution of each of these substances to heat conduction.

K. Shirai presented a heat conduction model (Shirai, 1965). In his model, air flow, vapor flow caused by air flow and diffusion, changes of water content and conduction of mineral particles played their respective shares of the role of heat conduction in soil. As his model includes almost all of the phenomena that are supposed to take part in the transfer of energy, so the result is very complicated, and it is difficult to come to a definite conclusion.

In the present paper, there is developed a mathematical model of heat conduction is porous media which is concerned with the macroscopic phenomenon of heat conduction. The heat conduction model here developed is used for measurement of water velocity through soil by applying the sinusoidal heat input. The model includes two media, one is stationary medium, and the other flowing medium. Both of them include mineral particles, water and air. Flowing medium consists of the flowing elements. Stationary medium consists of the elements which do not flow under given condition. Therefore, the elements of both media may change places with each other when the cond tion changes. Under two hypotheses shown in the following section the heat conduction model becomes simple and similar to the model of a uniform substance.

Nomenclature

a	kg/m^3	density
c	J/(kg·deg)	specific heat
θ	deg	temperature
u	deg/s	input temperature
λ	J/(m·s·deg)	termal conductivity
q	J/(m^3·s)	input energy
Q	J/m^3	internal energy
$\boldsymbol{v}$	m/s	velocity of flowing medium
$\boldsymbol{J}$	J/(m^2·s)	energy effluence density
$\boldsymbol{n}$		unit vector normal to the surface S
S	m^2	closed surface
ds	m^2	element of S
S_i	m^2	surface occupied by i
ds_i	m^2	element occupied by i
V	m^3	volume bounded by the closed surface S
dv	m^3	element of V
V_i	m^3	volume occupied by i
dv_i	m^3	element occupied by i
n_i		surface porosity
m_i		volume porosity
Subscripts		
$i=p, f$		
p		stationary part
f		flowing part

2) Hypotheses on porous media

When water moves through porous soil, heat is transferred directly by a flowing medium and indirectly by conduction in the stationary medium as well as in the flowing medium. To formulate the heat conduction by a simple energy equation, the writer wishes to propose two hypotheses, in relation to a flowing medium and a stationary medium, as follows:

Hypothesis I: Volume porosity m_i and surface porosity n_i are uniform everywhere in porous media,
where

$$m_i = \frac{V_i}{V}, \quad n_i = \frac{S_i}{S} \tag{1}$$

S: total area of closed surface of a soil portion,
V: total volume of the portion bounded by S,

subscript i denotes the following meaning,

$$i=\begin{cases} f: & \text{flowing part of } V \text{ or } S, \\ p: & \text{stationary part of } V \text{ or } S, \end{cases}$$

and

$$V = V_f + V_p, \quad S = S_f + S_p \tag{2}$$

This hypothesis means that the ratio of a flowing medium to a stationary medium will be constant at any infinitesimal portion.

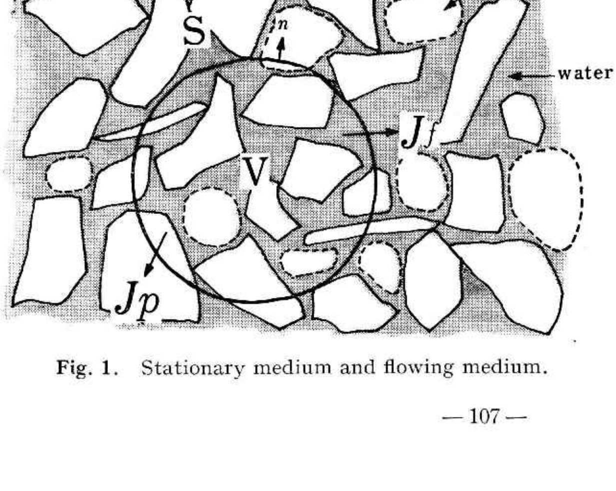
Hypothesis II: At any infinitesimal portion, the temperature of a flowing medium and that of a stationary medium are the same.

According to hypothesis I, a stationary medium and a flowing medium are well mixed at any infinitesimal portion, so that the conductivity is isotropic, and the distribution of temperature is continuous at any finite portion. This implies that at infinitesimal portions the temperature can be defined as the same in both of the media, flowing and stationary.

3) Equation of heat conduction

Energy equation in a portion of soil which consists of a stationary medium and a flowing medium will be described as follows. Input energy is equal to the sum of the increase of internal energy, the effluence of energy by heat conduction and that by the flow. Supposing that V is a volume of the portion bounded by a closed surface S, the above relationship is expressed as

$$\int_V q\mathrm{d}v = \int_V \frac{\partial Q}{\partial t}\mathrm{d}v + \int_S \boldsymbol{J}\cdot\boldsymbol{n}\mathrm{d}s + \int_S Q\boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}s \tag{3}$$



Soil is classified into two portions: a stationary medium and a flowing medium. Flowing medium consists of the flowing elements; stationary medium consists of the elements which do not flow under given condition. V denotes the small volume bounded by the closed surface S. $\boldsymbol{J}_p$ and $\boldsymbol{J}_f$ are vectors of the energy effluence density defined on the stationary part and the flowing part of S, respectively. $\boldsymbol{n}$ is the unit vector normal to S.

Fig. 1. Stationary medium and flowing medium.

where q is the input energy ($J\cdot m^{-3}\cdot s^{-1}$), Q the internal energy ($J\cdot m^{-3}$), $\boldsymbol{J}$ the vector of the effluent energy by heat conduction ($J\cdot m^{-2}\cdot s^{-1}$), $\boldsymbol{n}$ the unit vector normal to S, and $\boldsymbol{v}$ the velocity of the flow. Figure 1 shows a schematic explanation of the relationship of the variables mentioned above.

a) Internal energy

Internal energy consists of two parts: the energy of the stationary medium and that of the flowing medium. Let Q be the sum of Q_f and Q_p, and dv be the sum of dv_f and dv_p. The following relation will be obtained:

$$Q\mathrm{d}v = Q_f\mathrm{d}v_f + Q_p\mathrm{d}v_p \tag{4}$$

The following equations are derived by hypothesis I

$$\mathrm{d}v_f = m_f\mathrm{d}v, \quad \mathrm{d}v_p = m_p\mathrm{d}v \tag{5}$$

The internal energy can be written by the temperatures,

$$Q_f = a_f c_f \theta_f, \quad Q_p = a_p c_p \theta_p \tag{6}$$

By hypothesis II the temperatures of the two media are the same,

$$\theta_f = \theta_p = \theta \tag{7}$$

Hence Eq. (4) is rewritten as

$$\begin{aligned} Q\mathrm{d}v &= a_f c_f \theta_f m_f \mathrm{d}v + a_p c_p \theta_p m_p \mathrm{d}v \\ &= \overline{ac}\theta\mathrm{d}v \end{aligned} \tag{8}$$

where

$$\overline{ac} = a_f c_f m_f + a_p c_p m_p \tag{9}$$

Then the increment of the internal energy with time, which is expressed by the first term on the right side of Eq. (3), becomes

$$\int_V \frac{\partial Q}{\partial t}\mathrm{d}v = \int_V \frac{\partial}{\partial t}(\overline{ac}\theta)\mathrm{d}v = \int_V \overline{ac}\frac{\partial Q}{\partial t}\mathrm{d}v \tag{10}$$

The last equality can be established only when $\overline{ac}$ is independent of time.

b) Effluence by heat conduction

Let $\boldsymbol{J}$ be a vector of the effluent energy density, and $\boldsymbol{n}$ be a unit vector normal to S. The effluent energy through ds is

$$\boldsymbol{J}\cdot\boldsymbol{n}\,\mathrm{d}s \tag{11}$$

ds consists of two parts, the flowing part and the stationary part. Let ds_f and ds_p be these parts, respectively. Eq. (11) will be expressed as

$$\boldsymbol{J}\cdot\boldsymbol{n}\,\mathrm{d}s = \boldsymbol{J}_f\cdot\boldsymbol{n}\,\mathrm{d}s_f + \boldsymbol{J}_p\cdot\boldsymbol{n}\,\mathrm{d}s_p \tag{12}$$

where $\boldsymbol{J}_f$ is the effluent energy density of the flowing medium and $\boldsymbol{J}_p$ is that of the stationary medium, which are not defined on ds_p and ds_f respectively. As $\boldsymbol{J}$ is proportional to the gradient of temperature, Eq. (12) is rewritten as

$$\begin{aligned} \boldsymbol{J}\cdot\boldsymbol{n}\,\mathrm{d}s &= (-\lambda_f\nabla\theta)\cdot\boldsymbol{n}n_f\mathrm{d}s + (-\lambda_p\nabla\theta)\cdot\boldsymbol{n}n_p\mathrm{d}s \\ &= -\bar{\lambda}\nabla\theta\cdot\boldsymbol{n}\,\mathrm{d}s \end{aligned} \tag{13}$$

where

$$\mathrm{d}s_f = n_f\mathrm{d}s,\quad \mathrm{d}s_p = n_p\mathrm{d}s \tag{14}$$

$$\bar{\lambda} = \lambda_f n_f + \lambda_p n_p \tag{15}$$

Eq. (14) is based on the hypothesis I. Eq. (13) is substituted for the second term on the right side of Eq. (3). The effluence of energy by heat conduction is expressed as

$$\int_S \boldsymbol{J}\cdot\boldsymbol{n}\,\mathrm{d}s = \int_S(-\bar{\lambda}\nabla\theta)\cdot\boldsymbol{n}\,\mathrm{d}s = \int_V \nabla\cdot(-\bar{\lambda}\nabla\theta)\mathrm{d}s \tag{16}$$

The last equality of Eq. (16) is based on Gauss' theorem.

c) Effluence by flow

Flow is observed only in the flowing medium as is defined above. The effluence of energy by the flow will be expressed as

$$Q\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}s = Q_f\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}s_f \tag{17}$$

Eq. (17) is rewritten by Eqs. (6), (7) and (14) as follows:

$$Q\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}s = a_f c_f\theta\boldsymbol{v}\cdot\boldsymbol{n}n_f\mathrm{d}s \tag{18}$$

By using Eq. (18), the third term on the right side of Eq. (3) is converted into the following equations:

$$\int_S Q\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}s = \int_S a_f c_f n_f\theta\boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}s = \int_V \nabla\cdot(a_f c_f n_f\theta\boldsymbol{v})\mathrm{d}v \tag{19}$$

d) Input energy

q lying in the term on the left side of Eq. (3) expresses the input energy applied to the portion. Total q is composed of two parts. One is the energy applied to the flowing medium and the other the energy applied to the stationary medium. Hence the input energy will be written as

$$q\mathrm{d}v = q_f\mathrm{d}v_f + q_p\mathrm{d}v_p \tag{20}$$

The terms of Eq. (20) can be replaced by the following equations.

$$q_f = a_f c_f u_f,\quad q_p = a_p c_p u_p \tag{21}$$

$$u_f = u_p = u \tag{22}$$

Then the increase of the input energy is expressed as

$$\begin{aligned} q_f\mathrm{d}v &= a_f c_f u_f m_f\mathrm{d}v + a_p c_p u_p m_p\mathrm{d}v \\ &= \overline{ac}u\mathrm{d}v \end{aligned} \tag{23}$$

The unit of the input temperature ($\mathrm{deg}\cdot\mathrm{s}^{-1}$) means that the input temperature u is given during the unit of time. The last equality of Eq. (23) is obtained by Eq. (9). The term on the left side of Eq. (3), which expresses the input energy, is rewritten as

$$\int_V q\mathrm{d}v = \int_V \overline{ac}u\mathrm{d}v \tag{24}$$

e) Energy equation

Combining Eqs. (10), (16), (19) and (24), we have the energy equation expressed by the temperatures. Under hypotheses I and II, the complete energy equation through porous media is rewritten as follows:

$$\int_V \overline{ac}u\mathrm{d}v = \int_V \frac{\partial}{\partial t}(\overline{ac}\theta)\mathrm{d}v + \int_V \nabla\cdot(-\bar{\lambda}\nabla\theta)\mathrm{d}v + \int_V \nabla\cdot(a_f c_f n_f \theta \boldsymbol{v})\mathrm{d}v \qquad (25)$$

Under the assumptions of time independence and uniformity of $\overline{ac}$, Eq. (25) is converted into the following differential equation:

$$u = \frac{\partial\theta}{\partial t} - \nabla\cdot\left(\frac{\bar{\lambda}}{\overline{ac}}\nabla\theta\right) + \nabla\cdot\left(\frac{a_f c_f n_f}{\overline{ac}}\theta\boldsymbol{v}\right) \qquad (26)$$

or

$$\frac{\partial\theta}{\partial t} = \nabla\cdot(\bar{K}\nabla\theta) - \nabla\cdot(H\theta\boldsymbol{v}) + u \qquad (27)$$

where

$$\bar{K} = \frac{\bar{\lambda}}{\overline{ac}}, \qquad (28)$$

$$H = \frac{a_f c_f n_f}{\overline{ac}} \qquad (29)$$

4) Conclusion and Remarks

The heat conduction model through porous media was developed under two hypotheses. The model is quite similar to a model of heat conduction through a uniform substance. If flow is slow and steady, the spatial relation between the stationary medium and the flowing medium would be invariant. Therefore, the invariability of the volume porosity and surface porosity in hypothesis I can be admitted in such cases. The uniformity of these values and the continuity of the hypothesis II can be attained when the size of soil particles is sufficiently small and uniform.

The model is developed at first for the measurement of water velocity through soil, using the sinusoidal heat input. As the idea of this model is universal, the model will be used not only in heat conduction through soil but also in technological use, e. g. conduction through three and more kinds of media.

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土中の水分移動量の測定
—多孔質媒体の熱伝導モデル—

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多孔質媒体中を流れる流体による熱伝導モデルが二つの仮定の下に導かれている．熱エネルギーを伝える媒体として流動媒体と静止媒体の二つが考慮されている．二つの仮定は，1) 体質間ゲキ率と表面間ゲキ率は有孔媒体のあらゆる部分で一定，および 2) 任意の微小部分において流動媒体と静止媒体の温度は等しい，である．結果は一つの物質による熱伝導モデルに類似している．このモデルの概念は一般的であるので，広い応用が期待される．

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